't Hooft anomalies of discrete gauge theories and non-abelian group cohomology

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Introduction

- Discrete gauge theories (Dijkgraaf-Witten theories) are mathematically well-defined quantum field theories
- Provide an interesting toy model
- Relation to short range entangled topological phases of matter (group cohomology classification, boundary states, toric code model,...)

Goal for today

Study 't Hooft anomalies of discrete gauge theories in the functorial approach to quantum field theory.

't Hooft anomalies

A field theory on a manifold M consists of

- A space of fields $\mathcal{F}(M)$
- An exponentiated action functional $\exp(iS)\colon \mathcal{F}(M) \longrightarrow U(1)$

Symmetry: $G \curvearrowright F(M)$ preserving $\exp(iS)$
For the corresponding quantum theory

$$Z(M) = \int_{\phi \in \mathcal{F}(M)} \exp(iS) \mathfrak{D}\phi$$

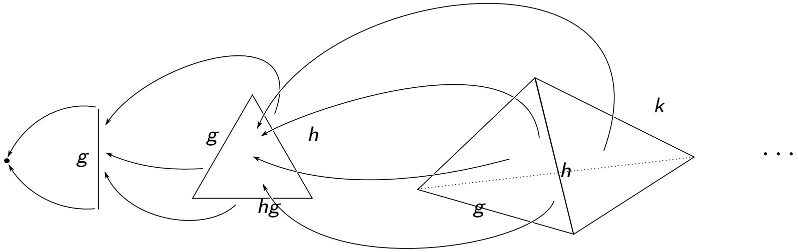
symmetries are realised by

- Invariants of $Z(M)$
- An action of G on the state space commuting with the Hamiltonian.

Gauging $\leftrightsquigarrow$ Coupling the theory to G-gauge fields
A 't Hooft anomaly is an obstruction to gauging the symmetry.

Discrete gauge theories

- Let D be a discrete group $\rightsquigarrow \mathfrak{d} = 0 \rightsquigarrow$ every D-bundle carries a unique (flat) connection
- D-bundles on $M \Leftrightarrow$ homotopy classes of continuous maps $M \longrightarrow BD$



Discrete gauge theories

Action for discrete gauge theories are classified by $\omega \in H^n(BD; U(1))$ [Dijkgraaf Witten '90]

$$\exp(2\pi i S(\varphi \colon M \longrightarrow BD)) = \int_M \varphi^* \omega = \langle \varphi^* \omega, [\sigma_M] \rangle$$

Quantum theory:

$$Z_{DW}(M) = \sum_{\varphi \in \pi_0(\mathrm{Bun}_D(M))} \frac{1}{|\mathrm{Aut}(\varphi)|} \int_M \varphi^* \omega$$

Reformulation as functorial field theory

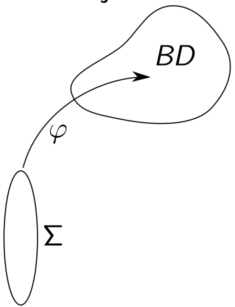
Definition

A *D-equivariant topological field theory* is a symmetric monoidal functor

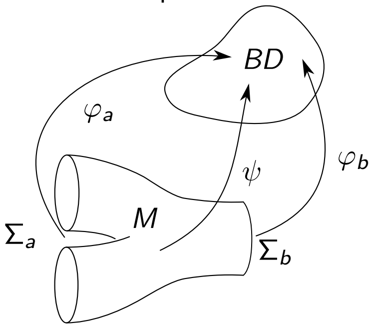
$$D\text{-Cob}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}} \ .$$

D-Cob_n:

Objects:

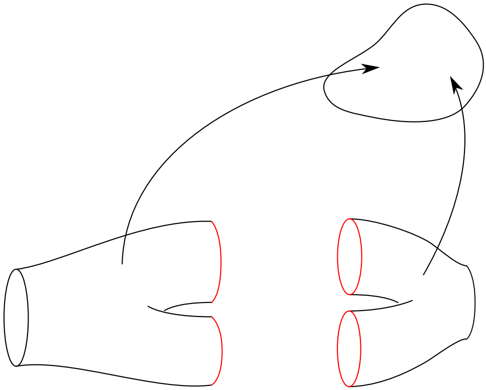


Morphisms:



Reformulation as functorial field theory

Composition:



Reformulation as functorial field theory

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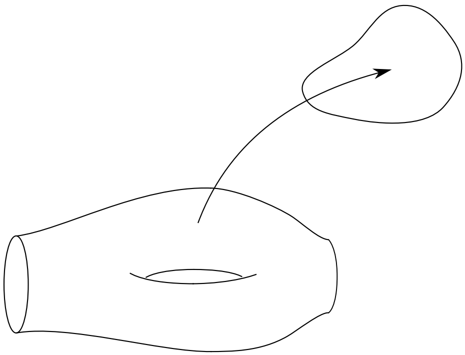

- Disjoint union defines the monoidal structure.

The classical theory $L_\omega \colon D\text{-Cob}_n \longrightarrow \text{Vect}_{\mathbb{C}}$ [Freed Quinn '93]

$$L_\theta(\Sigma, \varphi \colon \Sigma \longrightarrow BD) := \mathbb{C}[\{\sigma_\Sigma \in [\sigma_\Sigma]\}]/\sim_\omega$$

where $\sigma' \sim_\omega \langle \varphi^*\omega, \Lambda \rangle \sigma$ with $\partial\Lambda = \sigma' - \sigma$

$$\begin{aligned} L_\omega(M, \psi) : L_\omega(\Sigma_a, \varphi_a) &\longrightarrow L_\omega(\Sigma_b, \varphi_b) \\ [\sigma_a] &\longmapsto \langle \psi^*\omega, \sigma_M \rangle [\sigma_b] \ , \end{aligned}$$

with $\partial\sigma_M = \sigma_b - \sigma_a$

The quantum theory $Z_{DW} \colon \mathrm{Cob}_n \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ [Freed Quinn '93]

- Gauge transformations $h \colon (\varphi \colon \Sigma \longrightarrow BD) \Longrightarrow (\varphi' \colon \Sigma \longrightarrow BD) \leftrightsquigarrow h \colon [0,1] \times \Sigma \longrightarrow BD$
- This induces a map $L_\omega([0,1] \times \Sigma, h) \colon L_\omega(\Sigma, \varphi) \longrightarrow L_\omega(\Sigma, \varphi')$

$$Z(\Sigma) = \{\{f(\varphi) \in L_\omega(\Sigma, \varphi)\}_{\varphi \in \mathrm{Bun}_D(\Sigma)} \mid f(\varphi') = L_\omega(h) f(\varphi)\}$$

(This is $\lim_{\mathrm{Bun}_D(\Sigma)} L_\omega(\Sigma, \cdot)$)

Symmetries of discrete gauge theories

We consider symmetries coming from automorphisms of (the stack) $\mathrm{Bun}_D(\cdot)$

$$\mathrm{Aut}(\mathrm{Bun}_D) \cong \mathrm{Map}_{\mathrm{inv}}(BD, BD) \cong [\star /\!/ D, \star /\!/ D]_{\mathrm{inv}} \cong \mathrm{Aut}(D) /\!/ D$$

Symmetries only need to act up to gauge transformation $\rightsquigarrow$ we study homotopy coherent actions of G on $\mathrm{Bun}_D(\cdot)$

Proposition

There is a homomorphism of 2-groups $\mathrm{Aut}(\mathrm{Bun}_D) \longrightarrow \mathrm{Aut}(D\text{-Cob})$ *'*$(M, \psi\colon M \longrightarrow BD) \longmapsto (M, \psi\colon M \longrightarrow BD \longrightarrow BD)$*'.*

Definition

A homotopy coherent action ρ of G on $\mathrm{Bun}_D(\cdot)$ induces via pullbacks an action of G on $D-\mathrm{TFT}$. A *field theory with symmetry* ρ is a homotopy fixed point of this action.

Symmetries of discrete gauge theories

Proposition

Let ρ be a homotopy coherent action of G on BD. If ω is preserved in a coherent way then the classical Dijkgraaf-Witten theory L_ω admits a ρ-symmetry.

Definition

A theory with internal G symmetry is a symmetric monoidal functor $\mathrm{Cob}_n \longrightarrow G\text{-Rep}$.

Proposition

Every classical symmetry of L_ω induces a lift of Z_{DW} to a theory with internal symmetry $Z_{DW}\colon \mathrm{Cob}_n \longrightarrow G\text{-Rep}$.

Relation with non-abelian group cohomology

- A homotopy coherent action of G on BD can be described by a 2-functor

$$\alpha\colon \star /\!/ G \longrightarrow \star /\!/ \mathrm{Aut}(BD) \cong \star /\!/ \mathrm{Aut}(D) /\!/ D$$

- 2-functors $\star /\!/ G \longrightarrow \star /\!/ \mathrm{Aut}(D) /\!/ D$ are called non-abelian 2-cocycles
- They classify extensions

$$1 \longrightarrow D \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

Gauging symmetries

Definition

An internal symmetry of a field theory Z can be gauged if there exist a G-equivariant field theory $Z^G\colon G\text{-Cob} \longrightarrow \text{Vect}_{\mathbb{C}}$ such that the restriction to the trivial sector of Z^G agrees with Z.

Theorem (LM,R.J. Szabo '18)

Let Z_ω be a discrete gauge theory with topological action $\omega \in Z^n(BD; U(1))$ and kinematical G-symmetry described by an extension

$$1 \longrightarrow D \xrightarrow{\iota} \widehat{G} \xrightarrow{\lambda} G \longrightarrow 1$$

such that there exists $\widehat{\omega} \in Z^n(B\widehat{G}; U(1))$ satisfying $\omega = \iota^*\widehat{\omega}$. Then the symmetry can be gauged by the G-equivariant Dijkgraaf-Witten theory $\lambda_* L_{\widehat{\omega}}\colon G\text{-Cob} \longrightarrow \text{Vect}_{\mathbb{C}}$ [J. Maier T. Nikolaus C. Schweigert '11, LM L. Woike '18].

Anomalous boundary theories

- The obstructions for the existence of $\hat{\omega}$ are controlled by the Lyndon-Hochschild-Serre spectral sequence
- All obstructions but one vanish $\Rightarrow \exists \theta \in Z^{n+1}(BG; U(1))$ and $\hat{\omega} \in C^n(B\hat{G}; U(1))$ such that

$$\delta\hat{\omega} = \lambda^*\theta \text{ and } \iota^*\hat{\omega} = \omega$$

Theorem (LM, R.J. Szabo '18)

In this case the symmetry can be gauged as a field theory relative to the $n+1$-dimensional Dijkgraaf-Witten theory L_θ.

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Thank you for your attention!