NC Quantization of Geometry

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Over the last twenty years, we have managed to decipher the inner structure of space-time.

Working in the bottom to top approach we learned that, to good approximation, space-time is a product of a continuous four dimensional space tensored with a finite space.

. Spectral Data is

(A, H, D, γ, J)

Algebra, Hilbert Space, Dirac Operator, Chirality, Reality Operator

These satisfy certain relations

$$J^2 = \pm 1, \qquad JD = \pm DJ, \qquad J\gamma = \pm \gamma J$$

These relations define the KO dimension (mod 8)

Algebra of SM determined from bottom to top approach or
from classifying finite noncommutative spaces of is given by

$$\begin{aligned}
\mathcal{A} &= C^\infty(M) \otimes \mathcal{A}_F \\
\mathcal{H} &= L^2(S) \otimes \mathcal{H}_F \\
D &= D_M \otimes 1 + \gamma_5 \otimes D_F \\
\gamma &= \gamma_5 \otimes \gamma_F \\
J &= C \otimes J_F
\end{aligned}$$

Where

$$\mathcal{A}_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

Linearity of the connection restricts the algebra to a sub-algebra

$$\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

Dirac Operator for Finite Space comprise of Yukawa couplings and allow
for Dirac masses as well as Majorana mass for the right-handed neutrinos.

A new paradigm: Start with a relation representing the
fundamental class in KO homology in 2 dimensions

$$\langle Y [D, Y]^2 \rangle = \gamma$$

Where <> is the trace over the Clifford algebra defined by
the three 2 x2 Gamma matrices satisfying

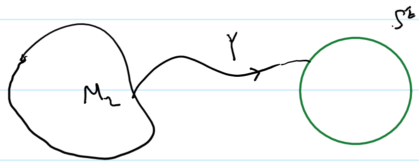
$$\begin{aligned}
\{\Gamma^A, \Gamma^B\} &= 2\delta^{AB}, \qquad A, B = 1,2,3 \\
Y^2 &= 1, \qquad Y = Y^* = Y_A \Gamma^A \\
\langle Y \rangle &= 0
\end{aligned}$$

This is a generalization of Heisenberg type relation where Y
are Clifford coordinates and D Feynman slashed momenta.

A local representation of the above relation takes the form

$$\epsilon_{ABC}\, Y^A dY^B dY^C = \sqrt{g}\, dx^1 \wedge dx^2$$

We note the following: The integral of the left-hand side is the winding number of the map from the two manifold to the two sphere, the right-hand side is the volume form. Thus this relation quantizes the volume of the two manifold in terms of the Planck volume of a two sphere.



Since det(g) $\neq$0, map Y is not singular, Jacobian does not vanish and topology of M must be that of a sphere, or is a disjoint collection of spheres giving rise to a bubble picture. This is undesirable.

The presence of the reality operator J offers a way out. Besides Y we can form Y'=JYJ commuting with Y.

$$Y' = i\, Y'^A \Gamma'_A \quad , \quad \{\Gamma'_A, \Gamma'_B\} = -2\,\delta_{AB} \quad , \quad Y'^2 = 1$$

$$Y \in M_2(\mathbb{C}), \qquad Y' \in \mathbb{H} \quad \text{quaternions}$$

To Modify Heisenberg like relation to accommodate both maps, we note that there is 1 to 1 correspondence between operators T^2=1 and projection operators e^2=e by T=2e-1

Let E=e, e', then Z=2E-1 and the modified relation is

$$\langle Z[D,Z]^2\rangle = \gamma$$

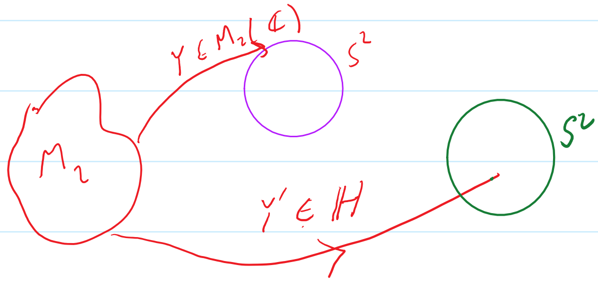
The surprise is that this formula factorizes to give

$$\langle Y[D,Y]^2\rangle + \langle Y'[D,Y']\rangle = \gamma$$

Whose local form is

$$\epsilon_{ABC}\left(Y^A dY^B dY^C + Y'^A dY'^B dY'^C\right) = \sqrt{g}\, dx^1 \wedge dx^2$$

Now, this relation does not necessarily imply that the Jacobian of the maps Y and Y' to be different than zero, but only the sum. We have the following picture:



deg(Y)+deg(Y')=N (winding number)

For a general map from a manifold of dimension n to the n-sphere, the ramification is of dimension n-2, and thus for the of two maps we must have (n-2)+(n-2)<n or n<4. Thus for n<4 it is possible to obtain an n-dimensional manifold with arbitrary topology from the inverse maps of Y and Y'.

Four dimensions is a critical case. We now address significance of the number 4

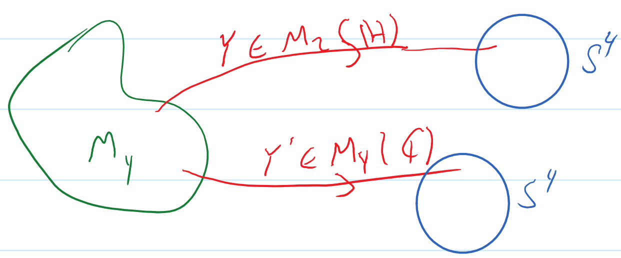
Our starting point is now the relation $T^4=1$ with the two solutions $T^2=1$ and $T^2=-1$. The let

$$\langle Z[D,Z]^4\rangle = \gamma$$

Where, as before, Z=2E-1, E=ee', Y=2e-1, Y'=2e'-1

$$\begin{aligned} Y &= Y^*, \qquad Y^2=1, \qquad Y=Y^A\Gamma_A, \qquad A=1,\cdots,5 \\ Y' &= Y'^*, \qquad Y'^2=1, \qquad Y'=iY^A\Gamma'_A, \\ \{\Gamma_A,\Gamma_B\} &= 2\delta_{AB}, \qquad \{\Gamma'_A,\Gamma'_B\}=-2\delta_{AB} \end{aligned}$$

The Clifford algebra for Y is $M_2(H)+M_2(H)$ and that of Y' is $M_4(C)$. By irreducibility, only one $M_2(H)$ is needed.



The ramifications of the maps Y and Y' are 2 dimensional surfaces which in general do intersect. We have shown that it is always possible to reconstruct four dimensional Riemannian spin-manifolds from the pullbacks of the maps Y and Y' provided that $\deg(Y)+\deg(Y') \leq 5$.

Miraculously the Heisenberg relation in terms of Z factorizes as the sum of Y and Y' terms with all interference terms vanishing. This property is not true for dimensions higher than four.

$$\langle Y[D,Y]^4\rangle + \langle Y'[D,Y']^4\rangle = \gamma$$

The local form of this equation is

$$\epsilon_{ABCDE}\left(Y^A dY^B dY^C dY^D dY^E + Y'^A dY'^B dY'^C dY'^D dY'^E\right)$$
$$= \sqrt{g}\, dx^1 \wedge dx^2 \dots \wedge dx^4$$

Given an element $Z(x)$ such that $Z^2=1$ and matrices m_i belonging to $M_2(H)+ M_4(C)$ then we can form words

$$a = \sum_i m_1 Z m_2 Z \dots m_i Z$$
$$m_i \in M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

The integral of this equation over the four manifold implies that the volume is quantized as a (sum) of integer multiple of the four sphere of Planck volume

This generate all spherical functions and thus the algebra A

$$C^\infty\left(M_4, M_2\left(\mathbb{H}\right)+M_4\left(\mathbb{C}\right)\right)=C^\infty\left(M_4\right)\otimes\left(M_2\left(\mathbb{H}\right)+M_4\left(\mathbb{C}\right)\right)$$

This is remarkable. Starting from a new paradigm, an equation representing fundamental classes of KO homology in four dimensions, in the form of an orientability condition, we arrived at the algebra of the noncommutative space obtained in the classification of finite spaces of KO dimension 6.

Elements of the algebra of the noncommutative space must commute with the chirality operator
$\gamma=\gamma_5\ \gamma_F$ with the later acting on $M_2(H)+M_4(C)$. This breaks $M_2(H)$ to H+H

$$\mathcal{A}_F = (\mathbb{H}_R \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})$$

The fundamental representation of the Hilbert space is then given by

$$\Psi = (\psi, \psi^c) \qquad \psi^c = C\psi^*$$

$$\psi_{\alpha q I} \quad : \quad \alpha \ \ M_4 \text{ spinor} \qquad \alpha = 1, \ldots 4$$

$$q : (\dot{a}, 1) + (1, a) \qquad \dot{a} = 1, 2 \in \mathbb{H}_R$$

$$a = 1, 2 \in \mathbb{H}_L$$

$$I : M_4(\mathbb{C}), \qquad I = 1, \ldots 4$$

This proves that the fundamental representation of the Hilbert space is a 16 dimensional space-time spinor, with the 16 components transforming under the representations of (2,1,4)+(1,2,4) with right-left $SU(2)_R \mathrm{x} SU(2)_L \mathrm{x} SU(4)_c$ symmetry. This is the Pati-Salam model where lepton number is the fourth color.

The free Dirac action is then simply given by

$$(\Psi, D\Psi)$$

$$J\Psi = \Psi$$

$$\gamma\Psi = \Psi$$

Satisfying both Majorana and Weyl conditions

The Dirac action is not invariant under inner automorphisms of the algebra A, but can be made so by adding a connection to the Dirac operator

$$\Psi \to U\Psi$$
$$U = u\hat{u}\,, \quad u \in \mathcal{A}, \quad \hat{u} = Ju^{*}J^{-1} \in \mathcal{A}^{\circ}$$
$$D_A = D + A\,, \quad D_A \to U D_A U^{*}$$

The connection A is constructed as follows

$$A = \sum a\hat{a}\,[D, b\hat{b}]. \qquad a \in \mathcal{A}, \quad \hat{a} \in \mathcal{A}^{\circ}$$
$$= A_{(1)} + J A_{(1)} J^{-1} + A_{(2)}$$

Here $A_{(1)}$ is linear while $A_{(2)}$ is not

$$A_{(1)} = \sum a\,[D, b] \qquad a, b \in (\mathbb{H}_R \oplus \mathbb{H}_L) \oplus M_4(\mathbb{C})$$

$$A_{(2)} = \sum \hat{a}\,[A_{(1)}, \hat{b}]$$

The linear connection $A_{(1)}$ includes the $SU(2)_R x SU(2)_L x SU(4)_c$ gauge fields as well as scalar fields in the $(2_R,2_L,1+15)$ representations while $A_{(2)}$ includes only scalar fields. This gives a Pati-Salam model.

A very important special case occurs when the order one differential constraint is satisfied by the algebra and Dirac operator [a,[D,b]]=0

This constraint reduces the algebra from

$$\mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C}) \longrightarrow \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

Which is that of the Standard Model. The connection A now will include the U(1)xSU(2)xSU(3) gauge fields in addition to one complex Higgs doublet (1,2,1). The connection $A_{(2)}$ reduces to a singlet scalar field which provides a Majorana mass for the right-handed neutrino.

Dynamics of the bosonic fields is governed by the spectral action principle which states that the spectrum of the Dirac operator D_A are geometric invariants,

$$I = Tr\left(f(D_A/\Lambda)\right)$$

Where f is a positive function which is additive for disjoint sets. This action reproduces all the details of the SM and is accord with experiment, thanks to the singlet field, up to very high energies. There is evidence, due to the lack of the meeting of the three gauge coupling constants at unification, that at some scale of the order of 10^{11} Gev, the Pati-Salam model becomes relevant.

We have thus shown that there are two kinds of quanta, in the form of sphere maps associated with two kinds of Clifford algebras. One of these algebras is associated with the four colors, and the other algebra corresponds to the right-left symmetry. These we refer to as Quanta of Geometry

In this picture we have seen that the fundamental fields are Y and Y' combined in the form of the field Z.
The gauge and Higgs scalar fields are constructed out of the field Z, and thus Z can be considered as a fundamental field and quantized. The metric, whose volume is quantized, can be constructed out of solitonic solutions.

$N=1$ $\qquad g_{\mu\nu} = \dfrac{2\,\delta_{\mu\nu}}{(1+x^2)^2} \qquad x^2 = x^a x^a, \quad a = 1,\dots,4$

Solitonic $\qquad Y^a = \dfrac{2x^a}{(1+x^2)}, \quad Y^5 = \dfrac{x^2-1}{x^2+1}$

$$\int \sqrt{g}\, d^4x = 1.\left(\frac{\pi^2}{2}\right)$$

N

$$g_{\mu\nu} = \frac{2\left(\partial_\mu X^N \partial_\nu \bar{X}^N + \partial_\nu X^N \partial_\mu \bar{X}^N\right)}{\left(1 + X^N \bar{X}^N\right)^2}$$

$$Y = \frac{2X^N}{(1 + X^N \bar{X}^N)} = Y^4 1 + Y^i e_i$$

$$Y^5 = \frac{X^N \bar{X}^N - 1}{(1 + X^N \bar{X}^N)}$$

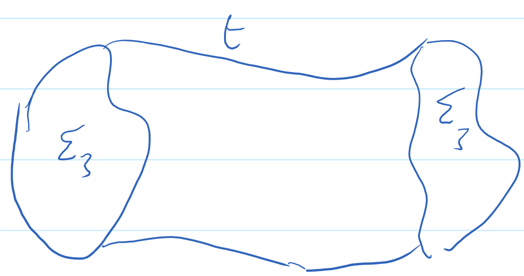
$e_1 e_2 = e_3$

$e_i^2 = -1$

1

$$\int \sqrt{g}\, d^4x = N\left(\frac{\pi^2}{2}\right)$$

We have thus far considered four dimensional Euclidean spaces with quantized volume. One would like to quantize three dimensional volumes and distinguish the time direction.



Distinguish one of the coordinates for each sphere, identifying them in the limit such that

$Y^5 = \eta X$ $\qquad$ $X^4 = \eta t$

$Y'^5 = \eta X$

$$g^{\mu\nu}\partial_\mu Y^5 \partial_\nu Y^5 = g^{\mu\nu}\partial_\mu Y'^5 \partial_\nu Y'^5 = 1$$

as $\eta \to 0$ $\qquad$ $g^{\mu\nu}\partial_\mu X \partial_\nu X = 1$

This is the mimetic constraint, and results in mimetic gravity, a minimal modification of gravity where the scale factor of the metric is exchanged with the mimetic field X. In synchronous gauge X=t. Mimetic gravity provides an explanation for dark matter, and can be used to construct cosmological models without the need for new fields.

Conclusion: We have made great progress so far providing a beautiful geometric picture and understanding for the fundamental forces and particles. We are finally in a very good position to tackle the problem of quantizing gravity and all matter in terms of more fundamental fields, the maps and their pullbacks from four manifolds to four spheres.