CONFORMAL BOOTSTRAP
"THEN AND NOW"

TEACHING THROUGH RESEARCH: REMEMBERING RAOUL

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GGI, FIORENCE 28 September 2018

The work of Raoul Gatto on the "CONFORMAL BOOTSTRAP" covered the period 1971 - 1974, in a collaboration with a small group at Frascati National Labs (CNEN) including Aurelio Grillo, Giorgio Parisi and myself -

In 1975 the collaboration ended with Raoul moving at the "University of Geneva", Giorgio moving to "Rome University", Grillo moving to the subject of Astrophysics and myself going to CERN..

A conference devoted to the subject of

"Scale and Conformal Symmetry in Hadron Physics"

organized by Gatto, took place at the

Frascati National Labs on MAY 1972.

(Book proceedings: Wiley-Interscience Publication, 1973)

At this conference results were presented

by several groups on diverse applications of

conformal symmetry. In particular our

main results on its application to short-distance

phenomena in relativistic quantum field theories.

These results covered the conformal covariant

OPERATOR PRODUCT EXPANSIONS (OPE), the embedding formalism

(DIRAC)
1936

and the crossing relations which are a consequence of locality, causality and associativity of the OPE's in the conformal setting: $\sum_{O} C_{AB}^{O}(x-y, \partial_y)\, O(y) = A(x) B(y)$

→ (fixed by CONFORMAL SYMMETRY)

Causality states that the conformal blocks of $A(x)B(y)$ are the same of $B(y)A(x)$ -

In particular note that $[A(x), B(y)] = 0 \quad (x-y)^2 < 0$

Associativity, which will be a dynamical constraint, states that

$$(\underbrace{A(x) B(y)}) C(z) = A(x) (\underbrace{B(y) C(z)})$$

(in block expansion)

This is not true block by block but

gives

1) $(A(x)\,B(y))\,C(z) = \boxed{\sum_O C^O_{AB}(x-y,\partial_y)\, O(y)}\, C(z)$

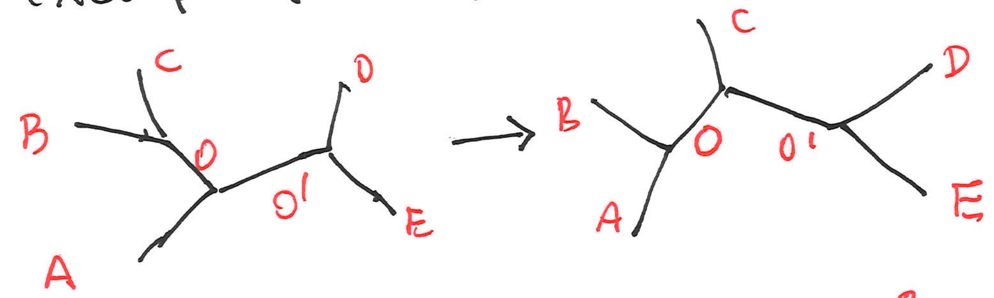
$= \boxed{\sum_{OO'} C^O_{AB}(x-y,\partial_y)\, C^{O'}_{OC}(y-z,\partial_z)\, O'(z)}$

2) $A(x)\,(B(y)\,C(z)) = \sum_O C^O_{BC}(y-z,\partial_z)\, A(x)\, O(z)$

$= \boxed{\sum_{OO'} C^O_{BC}(y-z,\partial_z)\, C^{O'}_{OA}(x-z,\partial_z)\, O'(z)}$

for example for a five-point function (or $n>4$)

2)



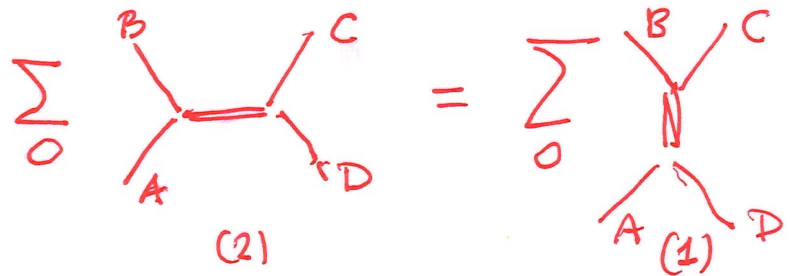
1)

(FGG Polyakov)

CROSSING RELATIONS

for the four point function

(associativity)



O fixed

CONFORMAL BLOCK

The kernel (differential operator with infinite terms) is closely related to the three point function, in fact, by taking

$$\langle A(x)\, B(y)\, C(z) \rangle = C_{AB}^{C}(x-y, \partial_y) \langle C(y)\, C(z) \rangle$$

$$= C_{BC}^{A}(y-z, \partial_z) \langle A(x)\, A(z) \rangle$$

$$= \frac{C_{ABC}}{(x-y)^{2\,\frac{l_A+l_B-l_C}{2}}} \frac{1}{(y-z)^{2\,\frac{l_B+l_C-l_A}{2}}} \frac{1}{(x-z)^{2\,\frac{l_A+l_C-l_B}{2}}}$$

Work on conformal OPE's and CROSSING (BOOTSTRAP) RELATION, other than POLYAKOV, was due to Mack, Todorov, Dobrev et al., Crewther, CERN, Ciccariello, Bonora, Sartori, Tonin (Padua), Parisi, Peliti

At the Frascati Conference (1972) Bardeen, Fritzsch, Gell-Mann, also based on previous work on the Light-Cone Current Algebra, presented results which relate the quark statistic to three different processes and which agree with experiments only with "Color" SU(3) and Free field theory at light cone distances (Bjorken scaling observed at SLAC). The other two processes being the total cross section $e^+e^- \to X$ at high energy and the $\pi^0 \to 2\gamma$ decay all related to OPE's of different currents

CONFORMAL SYMMETRY FOUND NEW IMPORTANT APPLICATIONS WITH THE ADVENT OF SPACE-TIME SUPERSYMMETRY (Wess, Zumino) AND ITS LAGRANGIAN REALIZATION. (1974 on)

Superconformal field theories with $N=1, 2$ superconformal supersymmetry were discovered and classified (N. Seiberg) (super Yang-Mills theories with matter multiplets) (S.F., Zumino, Salam, Strathdee)

Non renormalization theorems allow these theories to have exceptional properties as the existence of non trivial "conformal fixed points". A remarkable example is the $N=4$ supersymmetric Yang-Mills theory which is superconformal at arbitrary coupling (in perturbation theory)

Even if the Conformal Bootstrap was quiescent for almost ten years it had a first resurrection by the work of Belavin, Polyakov, Zamolodchikov (1984) where it was exactly solved for some classes of 2D conformal field theories which find application in string theory.

The existence of exactly solvable CFTs is believed to be a property of 2D conformal algebra (Virasoro algebra) which is infinite dimensional -

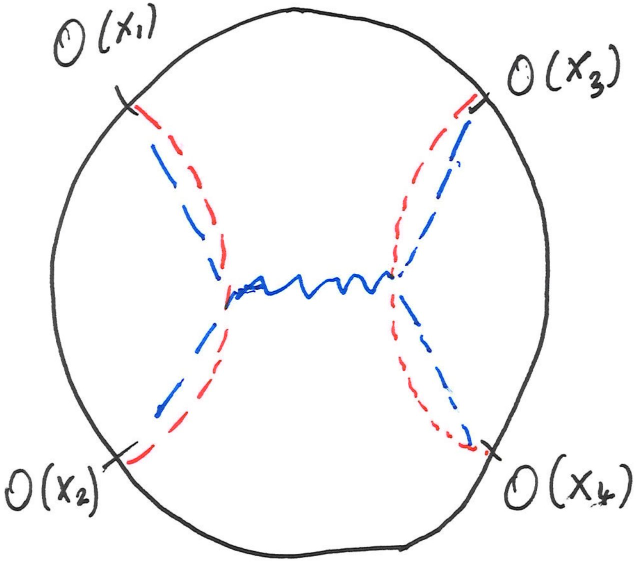
SAME applies for its superconformal extension when fermionic degrees of freedom are present in the worldsheet.

The CONFORMAL BOOTSTRAP program, namely the possibility of solving quantum field theories which are not perturbative neither lagrangean, was again resurrected in 2008 by the seminal work of Rattazzi, Rychkov, Tonni, Vichi "Bounding scalar operator dimensions in 4D CFT, JHEP, 12, 031 (2008) which opened the way to find new numerical and analytical methods to (approximately) solve the bootstrap (crossing) equations.

(See reviews of: D. Simmons-Duffin (arXiv:1602.07982; Feb 2016, D. Poland, S. Rychkov, A. Vichi (arXiv:1805.04405, May 2018) and L. Rastelli in: (Simons Foundation), program in: "Simons Collaboration on Non Perturbative Bootstrap. (Director of SC on NPB)

The conformal bootstrap program has made several advances in the last decade.

My personal view is the extension of "bootstrapping" to superconformal field theories with different number of N-extended supersymmetry and its role on the AdS-CFT correspondence where a mathematical relation between boundary and bulk amplitude is possible as well as an holographic description of the "conformal blocks" in term of Geodesic Witten Diagrams



Geodesic Witten Diagram

Geodesic: in AdS_{d+1} connecting the two boundary points (1–2, 3–4)

$$\underset{(x_1 x_2, x_3 x_4)}{CB} = \int_{\gamma_{12}} d\lambda \int_{\gamma_{34}} d\lambda' \, G_{b\partial}(y(\lambda), x_1)\, G_{b\partial}(y(\lambda), x_2)\, G_{bb}(y(\lambda), y(\lambda'), \ell, n)\, G_{b\partial}(y(\lambda'), x_3)\, G_{b\partial}(y(\lambda'), x_4)$$

(Integrated over a geodesic rather than the full bulk)

$\gamma_{12} \to y(\lambda)$

$\gamma_{34} \to y(\lambda')$

(Hijano, Kraus, Perlmutter, Snively)

HIGHLIGHTS OF THE FEATURE

EXPERIMENTAL INPUT : THE CASE FOR CONFORMAL SYMMETRY

CONFORMAL GROUP : GLOBAL ASPECTS

CONNECTED AND SIMPLY CONNECTED CONFORMAL GROUPS

EMBEDDING FORMALISM AND NOETHER THEOREMS , CASIMIR

TRANSFORMATIONS OF PRIMARY FIELDS AND UNITARITY BOUNDS

CORRELATION FUNCTIONS : CAUSALITY AND ASSOCIATIVITY

OPE'S TWO, THREE AND FOUR POINT FUNCTIONS

HYPERGEOMETRIC FUNCTIONS : LIGHT CONE AND S-CHANNEL OPE'S

$\underbrace{{}_1F_1 \,,\; {}_0F_1}_{\text{OPE}} \;;\; \underbrace{{}_2F_1 \,,\; F_4}_{\text{FOUR-POINT}}$

CONFORMAL BOOSTRAP , SHORT DINSTANCE, LIGHT, CONE , SPACE-LIKE

INFINITELY MANY PRIMARIES, LARGE DIMENSIONS AND SPIN

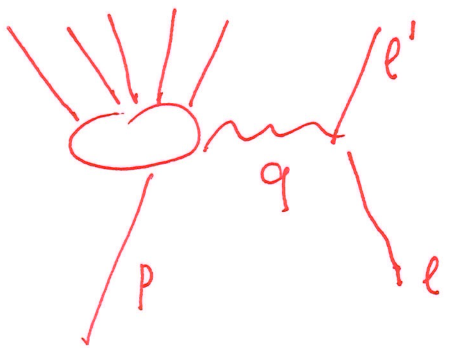
SLAC (late 60's/early 70's:

experiments in Deep Inelastic Scattering (DIS)

predict a "canonical" scaling of certain structure functions

which parametrize $e + p \rightarrow e + X$ (Bjorken Scaling, Feynman parton model)

(inclusive cross section lepton + proton → lepton + anything.



In the one-loop approximation the cross section depends on the correlator of two e.m. currents

$$W_{\mu\nu}(q,p) = \frac{1}{4\pi}\int d^4x\, e^{iqx} \langle p|\, J_\mu(x)\, J_\nu(0)\, |p\rangle$$

the scaling regime $\underset{\text{fixed}}{x} = \frac{-q^2}{2q\cdot p}$ ($q^2, q\cdot p$ large) is dominated

by $x^2 \to 0$ in the correlator. One can use OPE for

$$J_\mu(x)\, J_\nu(0) \quad x^2 \to 0$$

OPE

$$J_\mu(x)\,J_\nu(0) = \frac{C_0}{x^6}\left(\eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right) + \sum_n C^n_{\mu\nu}(x)\, x^{\alpha_1}\ldots x^{\alpha_n}\, O_{\alpha_1 \ldots \alpha_n}$$

$$+ \; C_{\mu\nu\rho}(x)\, J^{5\rho}(0) + \ldots$$

$$\langle p|\,J_\mu(x)\,J_\nu(0)\,|p\rangle = \frac{C_0}{x^6}\left(\eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right) + \sum_n C^n_{\mu\nu}(x)\, x^{\alpha_1}\ldots x^{\alpha_n} \langle p|\,O_{\alpha_1\ldots\alpha_n}(0)\,|p\rangle$$

$$= W_{\mu\nu}(x, p) \qquad \tau_n = \ell_n - n \sim 2$$

$$W_{\mu\nu}(q, P) = \frac{1}{4\pi}\int d^4x\, e^{iqx}\, W_{\mu\nu}(x, P)$$

$x = \dfrac{Q^2}{2q\cdot p}$, all kinematical variables in terms of q^2, $q\cdot P$, S

$Q^2 = -q^2$

$$S = (p+\ell)^2 = 2p\cdot\ell\,, \quad (P+q)^2 = \frac{1-x}{x}Q^2 + m_p^2\,, \quad y = \frac{P\cdot q}{P\cdot \ell} = \frac{Q^2}{x(S-m_p^2)}$$

$$Q^2 = xyS$$

OPE on the light-cone fixed by conformal OPE

$A(x)\ B(x)\ O_n(x)$

$l_A \quad l_B \quad l_n, n \quad \tau_n = l_n - n$

$$\underset{x^2\to 0}{A(x)B(0)} = \sum_{l_n,n} \frac{1}{(x^2)^{\frac{l_A+l_B-\tau_n}{2}}}\, x^{\alpha_1}\cdots x^{\alpha_n}\, {}_1F_1\left(\tfrac{1}{2}(l_A-l_B+l_n+n);\, l_n+n;\, x\cdot\partial\right) O_{\alpha_1\cdots\alpha_n}(0)\, C_n^{AB}$$

$$= (\text{coeff.}) \sum_{l_n,n} C_n^{AB} \frac{1}{(x^2)^{\frac{l_A+l_B-\tau_n}{2}}}\, x^{\alpha_1}\cdots x^{\alpha_n} \int_0^1 u^{\frac{1}{2}(l_A-l_B+l_n+n)-1} (1-u)^{\frac{1}{2}(l_B-l_A+l_n+n)-1}\, O_{\alpha_1\cdots\alpha_n}(ux)$$

Conformal Invariance fixes the OPE of two operators at finite distance $(x-y)^2$ finite.

Three or three operator product expansion:

$x-y\to 0$, $(x-y)^2\to 0$, $(x-y)^2$ finite (1971-1972)

Bjorken scaling implies the existence of infinitely many operators with twist $l_n - n = 2$. These operators are all leading on the LC and for $l_n - n = 2$, using conformal invariance and unitarity they are conserved $\partial^{\mu_1} O_{\mu_1 \dots \mu_{n-1}} = 0$ for $l_n = 2 + n$.

These are the symmetric traceless conformal primaries which exist in free-field theory. It is also in agreement with "asymptotic freedom" which assures that the conformal fixed point is the free field theory (zero coupling).

Conformal symmetry relate three processes

$$S \to \langle J^e(x) J^e(y) J^5(z) \rangle = S\, \Delta^{ee5}(x,y,z)$$

$$J^e(x) J^e(y) = R\, \Delta^{ee}(x,y)\, \mathbb{1} + K\, \Delta^{ee5}(x,y,\partial_y)\, J^5(y)$$

$$J^5(x) J^5(z) = R'\, \Delta^{55}(x,y)\, \mathbb{1} + \ldots$$

$$\langle J^e(x) J^e(y) J^5(z) \rangle = K R'\, \Delta^{ee5}(x,y,\partial_z)\, \Delta^{55}(y,z) = K R'\, \Delta^{ee5}(x,y,z)$$

so $S \sim K R'$; $S = A(\pi^0 \to 2\gamma) \to \int d^4y\, d^4z\, \epsilon^{\mu\nu\rho\sigma} y_\mu z_\nu \langle J_\rho(y) J_\sigma(z)\, \partial^\gamma J^5_\gamma(0) \rangle$

$|A(\pi^0 \to 2\gamma)|^2 = 1$ quark statistics and 3 color (up to a common normalization factor)

$|A(\pi^0 \to 2\gamma)|^2 = 1/9$ 3 F.D. quarks

$\sigma(e^+e^- \to X) / \sigma(e^+e^- \to \mu^+\mu^-)$ $2/3$ 3 FD quarks, 2 quark statistic with 3 color

$R \to 4$

"CONFORMAL BOOTSTRAP: THEN AND NOW"

P. Dirac (1936)

L. Castell (1966-68)

H. Kastrup, I. Todorov (1966)

Flato, Sternheimer (1966)

G. Mack, A. Salam (1969); G. Mack (1977)

Migdal / A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov (1984)

V.K. Dobrev, V.B. Petkova (1985)

S.F. R. Gatto A. Grillo (Annals of Physics 76 (1973) 161

A. Polyakov (1974), Zh. Eksp. Teor. Fiz 66 (1974)

S.F., R. Gatto, A. Grillo, G. Parisi (1972) F. Dolan, H. Osborn (2001)

R. Rattazzi, V.S. Rychkov, E. Tonni, A. Vichi (2008) 23

David Simmons-Duffin TASI Lectures (2016)

D. Poland, S. Rychkov, A. Vichi (2018)

CONFORMAL ALGEBRA ($SU(2,2) \sim SO^{+}(4,2)$)

$$[J_{AB}, J_{CD}] = i(\eta_{AB} J_{BC} + \eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC})$$

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{5\mu} = \tfrac{1}{2}(K_\mu - P_\mu), \quad J_{6\mu} = \tfrac{1}{2}(K_\mu - P_\mu), \quad J_{56} = D$$

$$C_{I} = J^{AB} J_{AB}, \quad C_{II} = \epsilon_{ABCDEF} J^{AB} J^{CD} J^{EF}, \quad C_{III} = J_A^{\ B} J_B^{\ C} J_C^{\ D} J_D^{\ A}$$

$\downarrow$ (in D dimensions $\to O(D,2)$) $D \geq 3$

$$\boxed{\begin{array}{ll} [P_\mu, K_\nu] = 2i(g_{\mu\nu} D - M_{\mu\nu}) & \\ [K_\mu, K_\nu] = 0 & [D, K_\mu] = i K_\mu \\ [P_\mu, P_\nu] = 0 & [D, P_\mu] = -i P_\mu \end{array}}$$

[Stability subalgebra at $x=0$ $D, M_{\mu\nu}, K_\mu$; finite (primary fields $K_\mu = 0$]

The conformal group (4,2) has four connected components, the same as the Lorentz group or any $O(p,q)$ group ($pq \neq 0$). (p space q time).

$SO(p,q) \rightarrow$ Special Conformal group Matrix Λ $\det \Lambda = 1$ is two (connected) components p,q orientation not reversed

$SO^{+}(p,q)$ or (p,q) orientations both reversed or both not reversed.

The other two connected components have $\det \Lambda = -1$ and correspond to reverse the q orientation or the p orientation, but not both. The four connected components are obtained by multiplying the matrices $L^{\uparrow}_{+}$ of $SO^{+}(p,q)$ with three matrices I_p, I_q, $I_{p,q}$ ($I_p^2 = I_q^2 = I_{pq}^2 = (I_p I_q)^2 = 1$). Which allow to define 4 subgroups of $O(p,q)$.

Component connected to the identity: $L_+^\uparrow = SO^+(p,q)$ $\det\Lambda = 1$

$\boxed{\text{Special orthogonal group: (Streater–Wightman)} \quad \text{Orthochronous orthogonal group} \quad \text{Orthochronous orthogonal group}}$

$\Lambda^t \eta \Lambda = \eta$ $\det\Lambda = 1$ $I_{pq} L_+^\uparrow + L_+^\uparrow = L_+$

$I_{pq}\Lambda$ $L_+^\downarrow$

$\det\Lambda = -1$ $I_p L_+^\uparrow + L_+^\uparrow = L^\uparrow$

$I_p\Lambda$ $L_-^\uparrow$

$\det\Lambda = -1$ $I_q L_+^\uparrow + L_+^\uparrow = L_0$

$I_q\Lambda$ $L_-^\downarrow$

$O(D,2)$ has a natural action on the embedding space (DIRAC) $E_{(D,2)}$ but if we want to make a manifest action on $M_{3,1}$ Minkowski space we must get rid of 2 coordinates. One goes to the $E_{(D,2)}$ light-cone and then identify points x_μ in space time, with rays $\eta_A = \lambda\eta_A$ on the $(D+2)$-dimensional light-cone (Mack, Salam)

EMBEDDING FORMALISM

The best way to obtain a (finite) conformal (K_μ boosts of parameter c_μ) transformation is to use the $(D,2)$ cone parametrized as follows $(D=4)$:

$$\boxed{\eta_\mu = k x_\mu \,, \quad \eta_5 + \eta_6 = k \,, \quad \eta_5 - \eta_6 = k x^2}$$

$\eta_\mu \; (1,1,1,-1)$

$\eta_5 \; (-1) = -\eta^5$

$\eta_6 \; (+1) = \eta^6$

Performing a L_{AB} rotation on $\eta_A = (\eta_\mu, \eta_5, \eta_6)$ we get (for $(AB) = (\mu 5, \mu 6)$): $L_{\mu 5}, L_{\mu 6} \to a_\mu, c_\mu$

$$\delta \eta_\mu = a_\mu k + c_\mu k x^2$$

$(a_\mu = \Lambda_{5\mu} - \Lambda_{6\mu}/2, \; c_\mu = (\Lambda_{5\mu} + \Lambda_{6\mu})/2$

$$\delta(\eta_5 + \eta_6) = 2 c_\mu k x^\mu$$

$$\delta(\eta_5 - \eta_6) = 2 a_\mu k x^\mu$$

so setting $a_\mu = 0$ we get for c_μ

$$\delta \eta_\mu = c_\mu k x^2 \,, \quad \delta k = 2 c_\mu k x^\mu \,, \quad \delta(k x^2) = 0$$

$$\delta x_\mu = \delta(\eta_\mu / k) = \delta\eta_\mu / k - \eta_\mu \frac{\delta k}{k^2} = c_\mu x^2 - 2 x_\mu \, x \cdot c$$

Now we use the fact that c_μ (as a_μ) is a nilpotent generator so its finite transformation on η_μ is the same as an infinitesimal one

$$\eta'_\mu = \eta_\mu + c_\mu(\eta_5 - \eta_6) = \eta_\mu + c_\mu K x^2 = K(x_\mu + c_\mu x^2)$$

$$\eta'_5 - \eta'_6 = \eta_5 - \eta_6 \rightarrow K' x'^2 = K^2 x^2$$

So we get $\eta'^\mu \eta'_\mu = K'^2 x'^2 = K^2 x^2 (1 + 2cx + c^2 x^2)$

$$K' x'^2 = K^2 x^2$$

Then

$$K' = K(1 + 2cx + c^2 x^2)$$
$$x'^2 = x^2 (1 + 2cx + c^2 x^2)^{-1}$$

$$\eta'_\mu = K' x'_\mu = K(x_\mu + c_\mu x^2)$$
$$x'_\mu = (x_\mu + c_\mu x^2)/(1 + 2c \cdot x + c^2 x^2)$$

and in the infinitesimal we retrieve

$$\xi_\mu = \delta x_\mu = c_\mu x^2 - 2 x_\mu x \cdot c$$

Noether Currents of space-time symmetries

$\partial^\mu(\xi_\rho \theta^\rho_\mu)=0$

$\partial^\mu \theta_{\mu\rho}=0$

$\theta_{\rho\mu}=\theta_{\mu\rho}$

$\theta^\lambda_\lambda=0$

The above is a particular solution of

$$\frac{1}{2}(\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) = \frac{1}{D}\eta_{\mu\nu}\partial^\rho \xi_\rho$$

Adding and subtracting $\frac{1}{2}\partial_\nu \xi_\mu$ we obtain

$$\partial_\nu \xi_\mu = \frac{1}{2}(\partial_\nu \xi_\mu - \partial_\mu \xi_\nu) + \frac{1}{D}\eta_{\mu\nu}\partial^\rho \xi_\rho$$ at parameter c_μ

which shows that the conformal transformation is a combination of an x-dependent Lorentz transformation and an x-dependent dilatation (preserve angles)

$$\frac{1}{2}(\partial_\nu \xi_\mu - \partial_\mu x_\nu) = 2(x_\nu c_\mu - x_\mu c_\nu)$$

$$\partial^\rho \xi_\rho = -2D\, x\cdot c \to \frac{1}{D}\eta_{\mu\nu}\partial^\rho\xi_\rho = -2\eta_{\mu\nu}\, x\cdot c$$

$$\partial_\nu \xi_\mu = 2(x_\nu c_\mu - x_\mu c_\nu) - 2\eta_{\mu\nu}\, x\cdot c$$

The above is the infinitesimal version of the ~~cho~~ Jacobian transform

$\frac{\partial x'^\mu(x,c)}{\partial x^\nu}$ which reads $\boxed{\frac{\partial x^\mu(x,c)}{\partial x_\nu} = \Omega(x,c)\, L(x,c)}$

with $\underline{\Omega(x,c)} = (1+2c\cdot x+c^2x^2)^{-1}$, $\underline{L^\mu_\nu} = \left[(\delta^\mu_\nu + 2c^\mu x_\nu) - \frac{2(x^\mu+c^\mu x^2)(c_\nu+x_\nu c^2)}{1+2c\cdot x+c^2x^2}\right]$

This is the transformation of $SO^+(p,q)$ $(p,q)=(4,2)$

written on a finite action on the cone rays has an x-dependent

(finite) dilatation and $SO^+(p-1,q-1)$ Lorentz transformation) .

The 4 connected components are implemented with

the I_t, I_s, I_{ts} transformations when I is the "inversion"

$$\boxed{x'_\mu = x_\mu/x^2 \;,\; x'_\mu = -x_\mu/x^2 \;,\; \text{with } I_s I_p = I_{sp} = -1}$$

note that I_s, I_p have det -1 while -1, 1 have det 1.

They correspond to space and time reversal of the orientation

in the $O(4,2)$ action on the cone $\eta_5 \to \eta_5$, $\eta_6 = -\eta_6$; $\eta_5 \to -\eta_5$ $\eta_6 \to \eta_6$

For the Inversion the corresponding Lorentz transformation is

$$\frac{\partial\, x^\mu/x^2}{\partial x^\nu} = \frac{1}{x^2}\left(\delta^\mu_\nu - 2\,\frac{x^\mu x_\nu}{x^2}\right) = \frac{1}{x^2}\, I^\mu_\nu(x)$$

Note that, unlike $L(x,c)$ (which for $c=0$ reduces to $\mathbb{1}$)

I, $-I$ belong to $L^\uparrow_+$ and $L^\downarrow_-$ i.e. are reflection on the time and space directions respectively.

One can easily check that the following relation follows

$$(x'-y')^2 = \frac{(x-y)^2}{(1+2c\cdot x+c^2x^2)(1+2c\cdot y+c^2y^2)}$$

which is a consequence of the above relations and the cone (S.2e) relation

$$\eta_x \cdot \eta_y = -\frac{1}{2} K_x K_y (x-y)^2 \qquad (\eta_x^2 = \eta_y^2 = 0)$$

Note that x_μ is invariant under $K \to \lambda K$ which indeed shows that x_μ (4 comp) parametrizes a ray on the cone rather than a point. So to all fields on the cone we must impose to be an eigenstate of the (Euler) Dilatation operator $\eta^A \partial_A = K \frac{\partial}{\partial K}$ to define fields which depend on rays rather than points in six (D) dimensions

A primary operator $O(x)$ at $x=0$ is classified by the $x=0$ stability algebra $(M_{\mu\nu}, D, K_\mu)$. By having $K_\mu=0$ on $O(x)$ we see that a primary operator is classified by three quantum numbers, a (J_L, J_R) rep. of $SL(2,C)$ $(SO^+(3,1))$ and a real number (Dilat.)

In terms of these numbers we have (in terms of $A_1 = J_L(J_L+1)$, $A_2 = J_R(J_R+1)$), l

$$\boxed{\begin{aligned} C_I &= l(l-4) + 2(A_1+A_2) \\ C_{II} &= (l-2)(A_1-A_2) \\ C_{III} &= (l-2)^4 - 4(l-2)^2(A_1+A_2+1) + 16A_1A_2 \end{aligned}}$$

For $J_L=J_R=\frac{n}{2}$, l $\rightarrow C_I = l(l-4)+n(n+2)$, $C_{II}=0$ $C_{III} = \begin{matrix}[l(l-2)-n(n+2)] \\ [(l-2)(l-4)-n(n+2)]\end{matrix}$

and C_{III} vanishes for conserved tensors $l=2+n$

PRIMARY CONFORMAL FIELDS

(under K boosts)

$$[\mathcal{O}(x), K_\lambda]^{(\beta)}_{(\alpha)} = i\left[(2x_\lambda x\cdot\partial - x^2\partial_\lambda)\,\delta^{(\beta)}_{\xi\alpha\xi} - 2ix^\nu(\eta_{\lambda\nu}\Delta + \Sigma_{\lambda\nu})^{\xi\beta\xi}_{\xi\alpha\xi}\right]\mathcal{O}_{\xi\beta\xi}(x)$$

Unitarity Bounds $J_L J_R = 0 \to \ell \geq 1 + J_L$ $(J_L, J_R) \to \ell \geq 2 + J_L + J_R$

Bound saturations: $\ell = 1 + J_L \to$ massless fields

$\ell = 2 + n \to$ conserved tensors (twist 2)

For a finite transformation

$$\boxed{\begin{aligned} \mathcal{O}'^{\beta}_{\alpha}(x') &= \frac{1}{(1+\cancel{2}\,c\cdot x + c^2x^2)^{\ell_0}}\, S_\alpha{}^{\beta}(L(x,c))\,\mathcal{O}_\beta(x) \\ &\text{under inversion} \\ \mathcal{O}'(x') &= \frac{1}{(x^2)^{\ell_0}}\, S_\alpha{}^{\beta}(I(x))\,\mathcal{O}(x) \end{aligned}}$$

To get a (scalar) field defined on x_μ we impose a homogeneity condition on $\Phi(\eta)_{\eta^2=0}$:-

Using the fact that $\eta^A \partial_A = K \frac{\partial}{\partial k}$ is well defined on the cone:

$$\eta^A \partial_A \Phi(\eta) = \lambda \Phi_\lambda(\eta) \Rightarrow \Phi_\lambda(x, K) = K^\lambda \varphi_\lambda(x)$$

so that $\varphi(x) = K^{-\lambda} \Phi_\lambda(\eta)$ is a field on $M_{3,1}$ with dimension $l = -\lambda$. One can check that

$$M_{56} \varphi_l(x) = (i x^\nu \partial_\nu + l) \varphi_l(x)$$

with

$$M_{AB} = i(\eta_A \partial_B - \eta_B \partial_A)$$

$$\frac{1}{2} M_{AB} M^{AB} = l(l-D) \quad \text{(in D dimensions)}$$

CORRELATION FUNCTIONS IN THE "EMBEDDING FORMALISM"

MAIN IDEA: $\langle 0 | [\varphi(x_1) \cdots \varphi(x_n), K_\lambda] | 0 \rangle = 0$

and then use $[\varphi(x), K_\lambda]$ as given before.

To make things simple we consider correlators on points $x_i \rightarrow$ Rays on the (4,2) cone.

So we must impose the Euler-homogeneity condition and O(4,2) rotational invariance

On n-point functions, dependence on $\frac{n(n-1)}{2}$ scalar products $\eta_i \cdot \eta_j$ and n Euler conditions $\longrightarrow n(n-3)/2$ variables

($n=2,3$ no arbitrariness, $n=4$ two variables are arbitrary

functions of two ~~vario~~ conformal invariant variables $u = \frac{\eta_1 \cdot \eta_2 \; \eta_3 \cdot \eta_4}{\eta_1 \cdot \eta_3 \; \eta_2 \cdot \eta_4}$, $v = \frac{(\eta_1 \cdot \eta_4)(\eta_2 \cdot \eta_3)}{\eta_1 \cdot \eta_3 \; \eta_2 \cdot \eta_4}$

$n=2$ $\quad \langle 0| \Phi_1(\eta_1)\, \Phi_2(\eta_2) \ldots \Phi_n(\eta_n) |0\rangle = A_n \Rightarrow \eta^i \partial_i A_n = -l_i A_n +$ $O(4,2)$ invariance

$$\boxed{F_{AB}(\eta_1\cdot\eta_2) = F(K_1K_2(X_1-X_2)^2 = [K_1K_2(X_1-X_2)^2]^{-l} C_{AB}}$$

So $k_1\frac{\partial}{\partial k_1} = -l_1$, $k_2 \frac{\partial}{\partial k_2} = -l_2$ has a solution if $l_1 = l_2$

$n=3$

$$\boxed{F_{123}(\eta_1\cdot\eta_2, \eta_1\cdot\eta_3, \eta_2\cdot\eta_3) = C_{ABC}(\eta_1\cdot\eta_2)^{-\frac{1}{2}(l_1+l_2-l_3)}(\eta_1\cdot\eta_3)^{-\frac{1}{2}(l_1+l_3-l_2)}\,\eta_2\cdot\eta_3^{-\frac{1}{2}(l_2+l_3-l_1)}}$$

$n=4$

$$A_{1234}(\eta_1\cdot\eta_2\ \eta_1\cdot\eta_3\ \eta_1\cdot\eta_4, \eta_2\cdot\eta_4, \eta_3\eta_4\ \eta_1\cdot\eta_3) = (\eta_1\cdot\eta_2)^{-l_B}(\eta_1\cdot\eta_3)^{\frac{-l_C+l_D-l_A+l_B}{2}} \cdot$$

$$\cdot(\eta_1\cdot\eta_4)^{\frac{-l_A+l_B+l_C-l_D}{2}}(\eta_3\cdot\eta_4)^{\frac{-l_C-l_D+l_A-l_B}{2}} g(u,v), \quad \boxed{u = \frac{\eta_1\cdot\eta_2\ \eta_3\cdot\eta_4}{\eta_1\cdot\eta_3\ \eta_2\cdot\eta_4}, \ v = \frac{(\eta_1\cdot\eta_4)(\eta_2\cdot\eta_3)}{\eta_1\cdot\eta_3\ \eta_2\cdot\eta_4}}$$

for $l_A = l_B = l_C = l_D$ $\quad \boxed{A = [(X_1-X_2)^2(X_3-X_4)^2]^{-l_A} g(u,v)}$

cone limit $(X_1-X_2)^2 \to 0$ $\quad A \Rightarrow [(X_1-X_2)^2(X_3-X_4)^2]^{-l_A} g(u\to 0, v)$

CAUSALITY (Block by Block relation) $\quad x_1 \to x_2$ or $x_3 \to x_4$

$$\boxed{g(u,v) = g\left(\frac{u}{v}, \frac{1}{v}\right)}$$

$$A(x_1)\,A(x_2) \to A(x_2)\,A(x_1)$$

ASSOCIATIVITY (CROSSING SYMMETRY)

$$\boxed{v^\ell g(u,v) = u^\ell g(v,u)}$$

$x_1 \to x_3$ or $x_2 \to x_4$

(or $x_1 \to x_4$, $x_2 \to x_3$)

$$\big(A(x_1)\,A(x_2)\big)\,A(x_3) = A(x_1)\,\big(A(x_2)\,A(x_3)\big)$$

Basic conformal bootstrap, insert OPE and try to solve (large dimension, large spin with few operators)

(the exact result is an infinite sum)

OPE EXPANSION AND CONFORMAL SYMMETRY

The OPE expansion is an operator algebra relation which asserts that a product of two local operators at two separated points x, y of space time can be decomposed in an infinite sum of local operators at point y with most singular operator coefficients (lowest dimensional operators) at $(x-y)^2 \to 0$ (or $x \to y$).

short $x \to y$ distance

The result for $AB \to C$ is (consequence of scale-symmetry)

$$\boxed{A(x)\,B(y) \underset{x \to y}{\Rightarrow} \left(\frac{1}{x^2}\right)^{\frac{l_A + l_B - l_C}{2}} C(y)} \quad \left(\text{for } l_A = l_B,\ C \to \mathbb{1}\right) \quad \left(\left(\frac{1}{x^2}\right)^{l_A} \mathbb{1}\right)$$

light cone $(x-y)^2 \to 0$

For tensor (traceless tensor operator $O_{\alpha_1 \ldots \alpha_n}(y)$ with twist $\tau_n = l_n - n$)

$$A(x)B(0) \underset{\substack{x \to 0 \\ \text{or } x^2 \to 0 \\ (x-y)^2 \to 0}}{\longrightarrow} \left(\frac{1}{x^2}\right)^{\frac{l_A + l_B - \tau_n}{2}} x^{\alpha_1} \cdots x^{\alpha_n} \overset{l_n}{O}_{\alpha_1 \ldots \alpha_n}(0) \to \left(\frac{1}{x^2}\right)^{\frac{l_A + l_B - l_n}{2}} O_n(0)$$

($y = 0$ by translation invariance)

So while on the light cone operators with the same twist have the same singularity, at short-distances operators with lower dimension give most singular contribution -

On the light cone operators with lesser twist have most singular contribution. Identity has twist 0, conserved currents have twist 2 and so on.

On the light-cone derivative operators count since

$$O_{\alpha_1 \dots \alpha_n}(x) \longrightarrow (x \cdot \partial)^p O_{\alpha_1 \dots \alpha_n}(x)$$

have the same dimension.

Out of the light cone operators of the form $(x^2 \Box)^q O_{\alpha_1 \dots \alpha_n}(x)$ have also the same dimension. Conformal symmetry relate such kind of operators so it is possible to compute the expansion at x-y (not short distance separated) not short-distance separated but arbitrary -

The summation of infinite terms of the type $(x-y)\partial_y$, $(x-y)^2\Box_y$ for three operator $A(x)B(y) \to O(y)$ can be formally written as

$$A(x)\,B(y) = \sum_{O} C^{O}(x-y, \partial_y)\, O(y)$$

when $C^{O}(x-y, \partial_y)$ is a known differential operator whose knowledge is strictly connected to the three point function via

$$\langle A(x)\, B(y)\, O(z)\rangle = C^{O}_{AB}(x-y, \partial_y)\,\langle O(y)\, O(z)\rangle$$

$C^{O}_{AB}(x-y, \partial_y)$ can be computed by computing both side with K_λ or by using the Embedding Formalism and requiring $O(4,2)$ covariance on light-cone and homogeneity.

The kernel $C^{\mathcal{O}}_{AB}$ was computed in the early 70's for arbitrary scalar fields A, B of dim. l_A, l_B and a tensor O_n with spin n and dimension l_n.

For simplicity we consider here $l_A = l_B$ and $n=0$ for which we have

$$\boxed{C^{\mathcal{O}}_{AA}(x-y,\partial_y)\,\mathcal{O}(y) = \left(\frac{1}{(x-y)^2}\right)^{l_A-\frac{l_0}{2}} C^{\mathcal{O}}_{AA}\,\frac{\Gamma(l)}{\Gamma(l_0/2)\Gamma(l_0/2)}\int_0^1 d\lambda\, \lambda^{+l_0/2-1}(1-\lambda)^{+l_0/2-1}\,{}_0F_1\left[l_0-1;\ \frac{-(x-y)^2}{4}\lambda(1-\lambda)\Box_y\right]\mathcal{O}(\lambda x+(1-\lambda)y)}$$

In the light cone limit ${}_0F_1$ becomes a constant and we have the conformal light-cone expansion

$$\boxed{C^{\mathcal{O}}_{AA}(x-y,\partial_y)\,\mathcal{O}(y) \underset{(x-y)^2\to 0}{\propto} \left[\frac{1}{(x-y)^2}\right]^{l_A-l_0/2} {}_1F_1\left(l_0/2;\ l_0;\ (x-y)\partial_y\right)\mathcal{O}(y)}$$

where $\boxed{{}_1F_1\left(l/2; l; (x-y)\partial_y\right)\mathcal{O}(y) \propto \int_0^1 d\lambda\,[\lambda(1-\lambda)]^{l_0/2-1}\,\mathcal{O}(\lambda x+(1-\lambda)y)}$

extension to spin Dolan, Osborn

Valid for $(x-y)^2$ arbitrary

${}_1F_1 \rightarrow$ confluent hypergeometric function one can check that

$$\boxed{C^{O}_{AA}(x-y,\partial_y)\langle O(y)\, O(z)\rangle \;\propto\; C_{OAA}(x,y,z) \;\propto\; \left[\frac{1}{(x-y)^2}\right]^{l_A - l_O/2} \left[\frac{1}{(x-z)^2(y-z)^2}\right]^{l_O/2}}$$

we can write any insertion to n-point functions to reduce to a product of lower ones with exchange of operators. For the four point function we have three possible operator (OPE) expansions because causality and associativity require strong relations between the "partial wave" amplitudes.

Embedding Formalism for OPE

$$\boxed{A(\eta)B(\eta') = \sum_{l,n} E_{l,n,A,B}(\eta\cdot\eta')\, D^{(n)A_1\dots A_n}(\eta,\eta')\,\psi_{A_1\dots A_n}(\eta')}$$

up to c-numbers $c_{l,n,AB}$

$\eta^2 = \eta'^2 = 0$

$\eta^A \partial_A A = -l_A A$

$\eta'^B \partial'_B B = -l_B B$

$$E_{l,n,A,B}(\eta\cdot\eta') = (\eta\cdot\eta')^{-\frac{1}{2}(l_A + l_B - l_n + n)}$$

$$\boxed{\begin{aligned} D^{(n)A_1\dots A_n} &= \eta^{A_1}\dots\eta^{A_n} D^h(\eta,\eta') \\ D(\eta,\eta') &= \eta\cdot\eta'\,\Box'_6 - 2\eta\cdot\partial'(1+\eta'\cdot\partial') \end{aligned}}$$

(well defined at $\eta^2 = \eta'^2 = 0$)

$h = -\frac{1}{2}(l_A - l_B + l_n + n)$ so that (since D is homogeneous of grade 1 in K/K')

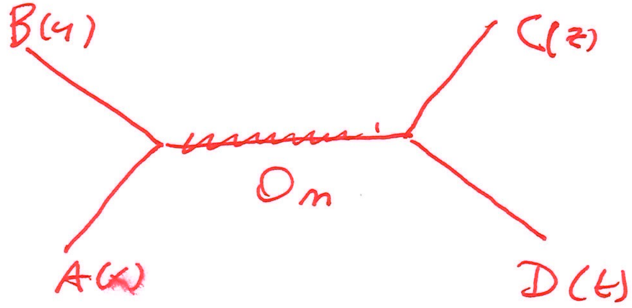
The right-hand side is homogeneous of degree $K^{-l_A - n} K'^{-l_B + l_m}$

which match the left hand side $\eta^{-l_A}\eta'^{-l_B}$ because of the factor $x^{A_1}\dots x^{A_n}\psi_{A_1\dots A_n}(x')$

with homogeneity degree $K^n K'^{-l_m}$ so remaining a $(\eta\cdot\eta')^{-\frac{l_A - l_B}{2}}$ factor s channel

$(\underbrace{A\ B})\ C$

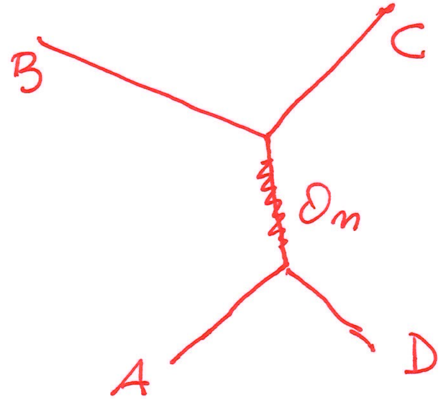
$$A(x)B(y) = \sum_{O} C^{O}_{AB}(x-y,\partial_y)\,O(y)$$



t channel

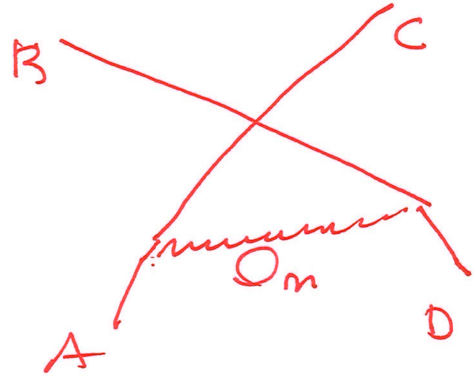
$A\ (\underbrace{B\ C})$

$$B(y)C(z) = \sum^{O} C^{O}(y-z,\partial_z)\,O(z)$$



u channel

$$A(x)C(z) = \sum^{O}_{AC} C^{O}(x-z,\partial_z)\,O(z)$$



$\underbrace{A\ B\ C}$

by multiplying now by D and taking VEV we obtain the CROSSING RELATION

$$\sum_{O} C^{O}_{AB}(x-y,\partial_y)\, C^{O}_{CD}(z-t,\partial_t)\, \langle O(y) O(t)\rangle = \qquad s$$

$$= \sum_{O} C^{O}_{AD}(x-t,\partial_t)\, C^{O}_{BC}(y-z,\partial_z)\, \langle O(t) O(z)\rangle = \qquad t$$

$$= \sum_{O} C^{O}_{AC}(x-z,\partial_z)\, C^{O}_{BD}(y-t,\partial_t)\, \langle O(z) O(t)\rangle \qquad u$$

These function can be calculated analytically to set for any ABCD tooOn exchange. Dory for ($\ell_A=\ell_B=\ell_C=\ell_D=\ell$) simplest, $A=B=C=D$ with a scalar O exchanged we have

$\langle A_1 A_2 A_3 A_4\rangle = \langle A(x_1) A(x_2) A(x_3) A(x_4)\rangle$ O exchange

LOCALITY + CAUSALITY (for each O_n) CROSSING SYMMETRY

$$g_{O_n}(u,v) = g_{O_n}\left(\frac{u}{v}, \frac{1}{v}\right) \qquad // \qquad v^{\ell_A} g(u,v) = u^{\ell_A} g(v,u)$$

KINEMATICAL ~~XXX~~ DYNAMICAL

The conformal block for the exchange of a scalar operator $\mathcal{O}(x)$ of dimension ℓ is obtained by inserting twice the OPE in the $x_1 x_2$ and $x_3 x_4$ channels. The resulting

$g_{\mathcal{O}_{\ell}}(u,v) \equiv g_{\mathcal{O}_{\ell}}(u,v)$ is in terms of a "double hypergeometric function F_4

with the following reduction formula $F_4(\alpha, \beta; \gamma; \gamma'; x=0, y) =$

$= {}_2F_1(\alpha, \beta; \gamma'; y)$; $\langle A(x_1) A(x_2) A(x_3) A(x_4) \rangle = \left[\frac{1}{(x_1-x_2)^2 (x_3-x_4)^2}\right]^{\ell_A}$

$$(F_4) \quad g_{\ell}(u,v) \propto \int_0^1 d\sigma [\sigma(1-\sigma)]^{-1} \left(\frac{v}{u\sigma} + \frac{1}{u(1-\sigma)}\right)^{-\ell_0/2} {}_2F_1\left(\tfrac{1}{2}\ell_0, \tfrac{1}{2}\ell_0; \ell_0 - 1; \left(\frac{v}{u\sigma} + \frac{1}{u(1-\sigma)}\right)^{-1}\right)$$

In the light cone limit $u \to 0$ (v fixed) $\quad g_{\ell}(u,v) \underset{u \to 0}{=} u^{\ell_0/2}\, {}_2F_1\left(\tfrac{1}{2}\ell_0, \tfrac{1}{2}\ell_0; \ell_0; 1-v\right)$

So the amplitude (with the $\mathcal{O}$ exchange), in the light cone limit is

$$A_4 \underset{(x_1-x_2)^2 \to 0}{\longrightarrow} \left[\frac{1}{|x-y|^2} \frac{1}{|z-t|^2}\right]^{\ell_A - \ell/2} \left[(x-z)^2 (y-t)^2\right]^{-\ell/2} {}_2F_1(\ell/2, \ell/2; \ell; 1-v)$$

So we have a hierarchy of hypergeometric functions which appear in different limits

1) OPE expansion as differential operators.

$_1F_1$, confluent hypergeometric function $x^2 \to 0$

F_1, generalized hypergeometric function x^2 finite

2) Conformal blocks

$_2F_1$ hypergeometric function $x^2 \to 0$

F_4 double hypergeometric function x^2 finite

The OPE depend on different variables which we can analytically continue. They are the space-time dimension and the primary quantum numbers, for example for symmetric traceless tensors we have $\tau_n = \ell_n - n$ (twist) and ℓ_n. The hypergeometric functions depend analytically in these parameters (D, ℓ_n, n)

Solution of conformal bootstrap equations

(crossing symmetry) for identical scalars A, B, C, D

$$g(u,v) = \sum_{O} f_{AAO}^2 \, g_{\ell_A, \ell_O} = \sum_{(\ell_m, n)} f_{AO_{\ell n}}^2 \, g_{\ell_A, \ell_m}$$

By using crossing symmetry for the $\langle A(x_1) A(x_2) A(x_3) A(x_4) \rangle$

amplitude we have $(\ell_A = \ell)$

$$\sum_{O_m} f_{AAO_m}^2 \left(v^{\ell} g_{A,\ell_m,n}(u,v) - u^{\ell} g_{A,\ell_m,n}(v,u) \right) = 0$$

for u, v finite.

This eq. can be regarded as a sum with positive coefficients

of an infinite dimensional vector V_x being zero

(Simmons-Duffin pg. 44-54)
uv eq ee

$$\sum_x \vec{V}_x \, c_x^2 = 0 \qquad \vec{V}_x = V_x(u,v)$$

INFINITE MANY PRIMARIES

Crossing symmetry with the unit operator requires infinite primaries on the right hand side of bootstrap eqs.

Indeed in the $x_1 x_2$ $(x_3 x_4)$ channel we have

$$A(x)\,A(y) = \frac{1}{(x-y)^{2\ell_A}}\,\mathbb{1} \qquad \langle A(x)A(y)A(z)A(t)\rangle = \left[\frac{1}{(x-y)^2}\,\frac{1}{(z-t)^2}\right]^{\ell_A} f_{\mathbb{1}}(u,v)$$

$f_{\mathbb{1}} = $ const

what is the operator with smallest dimension $\ell_{0=\mathbb{1}} = 0$ in a unitary conformal field theory.

Crossing the $u \leftrightarrow v$ (block) $\to$

$$g_0(v,u) = v^{\ell_0/2}\,{}_2F_1\left(\tfrac{1}{2}\ell_0, \tfrac{1}{2}\ell_0; \ell_0; 1-u\right) \to \log \text{ singularity for } u \to 0 \quad (v \to 1) \quad (x_1 \to x_2)$$

The amplitude goes as log for $u \to 0$

$$\cancel{A_4 \to \frac{1}{[(x_1-x_2)^2 (x_3-x_4)^2]^{\ell_A}}\, v^{\ell/2}\,{}_2F_1\left(\tfrac{1}{2}\ell, \tfrac{1}{2}\ell; \ell; 1-u\right)}$$

$$\cancel{v \to 1 \; (x_1 \to x_2)} \quad u \to 0$$

Using the crossing relation formula, if it would be true for any block it would imply

$$g_{0\ell}(u,v) = \left(\frac{u}{v}\right)^{\ell_A} g_{0\ell}(v,u)$$

The crossed block for $u \to 0$, $v \to 1$ gives a behaviour

$u^{\ell_A} \log u$ since $g_{0\ell}(v,u) = v^{\ell/2} {}_2F_1(\frac{1}{2}\ell_0, \frac{1}{2}\ell_0, \ell_0; 1-u)$

and for $u \to 0$ (v fuse on $v=1$ at short distances $g_0(u,v) \xrightarrow[u\to 0]{} \log u$) $(v \doteq 1)$ $(x_1 \to x_2)$

short distance

The expression $u^{\ell_A} \log u$ goes to zero $\forall\ \ell_A$ so one need an infinite series of primaries with large dimensions to reproduce $f_{\mathbb{1}}(u,v) = \text{constant}$. Making the limit $u \to 0$, v fixed (light-cone) one concludes that infinite spinning conformal blocks are needed with large spin

For spinning conformal blocks, when a tensor (symmetric, traceless) of rank n is exchanged we have (n even)

$$\langle A(x)\,A(y)\,A(z)\,A(t)\rangle \underset{(x-y)^2\to 0}{\simeq} \left[\frac{1}{(x-y)^2(z-t)^2}\right]^{\ell_A} u^{\tau_n/2}\, {}_2F_1\left(\tfrac{1}{2}d_n, \tfrac{1}{2}d_n; d_n; 1-v\right)$$

$$d_n = \ell_n + n\,, \quad \tau_n = \ell_n - n$$

so the 4 point funct behave as

$$A_4 \underset{(x-y)^2\to\infty}{\longrightarrow} \left[\frac{1}{(x-y)^2(z-t)^2}\right]^{\ell_A - \frac{\tau_n}{2}} \left[(x-z)^2(y-t)^2\right]^{-\tau_n/2} {}_2F_1\left(\frac{d_n}{2}, \frac{d_n}{2}; d_n; 1-v\right)$$

so for each conformal block

$g_{O_n}(u,v) \sim u^{\tau_n/2}(1+\dots)$ – So the ~~crossing~~ relation with the identity, to avoid the v dependence must take limits but in τ_n and d_n (on the light cone, one of the crossing ratios is eliminated $u\to 0$ v fixed)