Are locality and renormalisation reconcilable ?

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Brain teaser 1

What does the **harmonic sum**

$$S := "1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots" = \sum_{k=1}^{\infty} k^{-1}"$$

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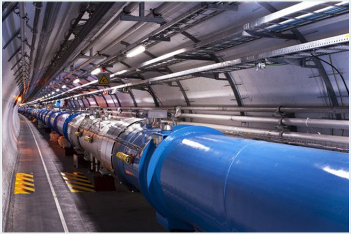

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Example

$\left(f_1(z) = z \wedge f_2(z) = \frac{1}{z}\right) \Longrightarrow f_1^{\rm reg}(0)\, f_2^{\rm reg}(0) = 0 \neq 1 = (f_1\, f_2)^{\rm reg}(0)$.

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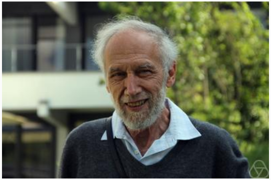

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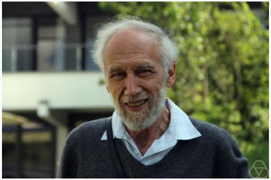

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A third multivariate approach

(with P. Clavier, L. Guo and B. Zhang)

using algebraic **locality**

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Analogy: separation of variables ($n = n_1 + n_2$)

$$\underbrace{\int_{\mathbb{R}^n} f_1(x_1)\, f_2(x_2)\, dx_1\, dx_2}_{x_1 \text{ and } x_2 \text{ independent}} = \underbrace{\left(\int_{\mathbb{R}^{n_1}} f_1(x_1)\, dx_1 \right) \cdot \left(\int_{\mathbb{R}^{n_2}} f_2(x_2)\, dx_2 \right)}_{\text{multiplicativity}}.$$

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- the coproduct on the source space $\mathcal{A}$ for a locality relation on the target space $\mathcal{M}$: $\Delta_{\mathcal{A}} \rightsquigarrow \top_{\mathcal{M}}$;
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LOCALITY

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Locality cont'd.

Local functionals in QFT

Functionals F on fields ϕ of the form $F(\phi) = \int_M f\left(j_x^k(\phi)\right)\, dx$, where $j_x^k(\phi)$ is the k-th jet of ϕ at x. Here, $\mathrm{Supp}\,(f\,(\psi)) \subset \mathrm{Supp}\,(\psi)$.

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Link between various concepts of locality

$\Psi^\Gamma_{\rm phg}(M)$ polyhomog. pseudodiff. operators on M with order in $\Gamma \subset \mathbb{C}$: A linear form $\Lambda : \Psi^\Gamma_{\rm phg}(M) \longrightarrow \mathbb{C}$ with $A \longmapsto \Lambda(A)$, is local if and only if $\mathrm{Supp}(\chi) \cap \mathrm{Supp}(\chi') = \emptyset \Longrightarrow \Lambda(\chi\, A\, \chi') = 0$.

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Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions $\phi, \psi \in \mathcal{D}(U)$ are independent i.e., $\phi \top \psi$ whenever $d\,(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)) > \epsilon$.

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For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

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Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

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Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \Longleftrightarrow m \wedge n = 1.$$

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Counterexample

Equip $\mathbb{R}$ with the locality relation $x \top y \Longleftrightarrow x + y \notin \mathbb{Z}$.
$(\mathbb{R}, \top, +)$ is NOT a locality semi-group: for $U = \{1/3\}$ we have
$(1/3, 1/3) \in (U^\top \times U^\top) \cap \top$ but $1/3 + 1/3 = 2/3 \notin U^\top$.

MULTIVARIATE GERMS

Brain teaser 2

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Evaluating a fraction with a linear pole at zero

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Evaluating a fraction with a linear pole at zero

$$\frac{z_1 - z_2}{z_1 + z_2}\Big|_{z_1=0, z_2=0} = \begin{cases} 1\,? \\ 0\,? \\ 10000\,? \end{cases}$$

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In our approach, a given choice of locality fixes the value 0.

Independent multivariate meromorphic germs

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Multivariate meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,

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Locality: separation of variables

On $\mathcal{M}(\mathbb{C}^\infty) = \bigcup_{k\in\mathbb{N}} \mathcal{M}(\mathbb{C}^k)$, $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

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$\ell := z_1 \perp z_2 =: L \Longrightarrow \frac{z_1}{z_2} \in \mathcal{M}_-(\mathbb{C}^2)$

$(\ell := z_1 - z_2) \perp (z_1 + z_2 =: L) \Longrightarrow \frac{z_1 - z_2}{z_1 + z_2} \in \mathcal{M}_-(\mathbb{C}^2)$.

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$z_1 - z_2 \perp \left(z_1 + z_2 \Longrightarrow \frac{z_1 - z_2}{z_1 + z_2} = z_1 - z_2 \cdot \frac{1}{z_1 + z_2} \right.$.

Multivariate decomposition theorem

Theorem (L. Guo, S.-P., B. Zhang/ N. Berline, M. Vergne 2015)

$\mathcal{M}(\mathbb{C}^k) = \mathcal{M}_-(\mathbb{C}^k) \oplus^\perp \mathcal{M}_+(\mathbb{C}^k)$, where $\mathcal{M}_-(\mathbb{C}^k) \ni \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$ with $\mathrm{Dep}(h) \perp \langle L_1, \cdots, L_n \rangle$ and $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

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$$f \longmapsto f^{\mathrm{reg}}(0) := \mathrm{ev}_0^{\mathrm{reg}}(f)$$

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RENORMALISATION and LOCALITY
reconciled

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$\Phi : (A, \top_A, m_A) \longmapsto (B, \top_B, m_B)$ is moreover a locality morphism of locality semi-groups if $a_1 \top_A a_2 \Longrightarrow \Phi(m_A(a_1, a_2)) = m_B(\Phi(a_1), \Phi(a_2))$.

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Our task

Build a locality character $\Phi^{\mathrm{reg}} : (\mathcal{A}, \top_A, m_A) \longrightarrow (\mathbb{C}, \cdot)$

$$a_1 \top_A a_2 \Longrightarrow \Phi^{\mathrm{reg}}\,(m_A(a_1, a_2)) = \Phi^{\mathrm{reg}}(a_1) \cdot \Phi^{\mathrm{reg}}(a_2). \qquad (1)$$

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Theorem

A locality morphism $\Phi : (\mathcal{A}, \top) \longrightarrow (\mathcal{M}(\mathbb{C}^k), \perp)$ gives rise to a locality character

$$\Phi^{\mathrm{reg}} := ev_0{}^{\mathrm{reg}} \circ \Phi : (\mathcal{A}, \top) \longrightarrow \mathbb{C}.$$

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Summary

A multivariate regularisation provides a renormalisation scheme which respects locality .

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3. Feynman amplitudes $(z_e, e \in \mathcal{E}(\Gamma)) \longmapsto \prod_e G(z_e)$, with $G(z_e)$ the kernel of $(\Delta + m^2)^{-1+z_e}$, on each edge e of the graph Γ.

Conclusions

One can renormalise at poles while preserving locality

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Open questions

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Group actions

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- How does it act on $\Phi^{\mathrm{reg}}(\mathcal{A})$?

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To appear (with P. Clavier, L. Guo and B. Zhang):

- Renormalisation via locality morphisms.
- Renormalisation and locality: branched zeta values.