Coherent states on some new fuzzy spheres

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Introduction

We have recently [Ref. below] introduced

- a fully $O(2)$-covariant fuzzy circle $\{S^1_\Lambda\}_{\Lambda\in\mathbb{N}}$,
- a fully $O(3)$-covariant fuzzy 2-sphere $\{S^2_\Lambda\}_{\Lambda\in\mathbb{N}}$.

We resp. start from a quantum particle in $\mathbb{R}^2,\mathbb{R}^3$ u a confining potential $V(r)$ with a very sharp minimum on the sphere of radius $r=1$ and impose a suitable energy cutoff; cutoff and sharpness $V''(1) =: 4k$ of the potential well parametrized by (and diverge with) Λ.
1^{st} motivation: alg. rel. covariant also under $x_i \mapsto -x_i$ ($\neq$Madore).

Here I wish to report on some further investigations about the geometry of these S^d_Λ:
Coherent States (CS);
Spectrum of the space coordinate operators x_i.

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$O(2)$-equivariant fuzzy circle - The essentials

We previously constructed the $O(2)$-equivariant fuzzy circle $\{\mathcal{A}_{\Lambda,2}\}_{\Lambda\in\mathbb{N}}$, it is a sequence of unitary irreducible representations $(\pi_{\Lambda,2},\mathcal{H}_{\Lambda,2})$ of $Uso(3)$ and every $\mathcal{A}_{\Lambda,2}$ acts on the corresponding Hilbert space

$$\mathcal{H}_{\Lambda,2} := span\,\{\psi_m | m \in \mathbb{Z}, -\Lambda \leq m \leq \Lambda\}.$$

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The noncommutative coordinates $x_+ := \frac{x_1+ix_2}{\sqrt{2}}$ and $x_- := \frac{x_1-ix_2}{\sqrt{2}}$ generate the $*$-algebra $\mathcal{A}_{\Lambda,2}$ and their actions read

$$x_+\psi_m = \begin{cases} \frac{b_{m+1}}{\sqrt{2}}\psi_{m+1} & \text{if } -\Lambda\le m\le\Lambda-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$x_-\psi_m = \begin{cases} \frac{b_m}{\sqrt{2}}\psi_{m-1} & \text{if } 1-\Lambda\le m\le\Lambda \\ 0 & \text{otherwise,} \end{cases}$$

where $b_m := \sqrt{1+\frac{m(m-1)}{k}}$.

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$L, x_+, x_-, \boldsymbol{x}^2$ fulfill the $O(2)$-covariant relations

$$[L, x_\pm] = \pm x_\pm, \quad {x_+}^\dagger = x_-, \quad (L)^\dagger = L, \tag{1}$$

$$[x_+, x_-] = -\frac{L}{k} + \left[1 + \frac{\Lambda(\Lambda+1)}{k}\right] \frac{\widetilde{P}_\Lambda - \widetilde{P}_{-\Lambda}}{2}.,\tag{2}$$

$$\boldsymbol{x}^2 = 1 + \frac{L^2}{k} - \left[1 + \frac{\Lambda(\Lambda+1)}{k}\right] \frac{\widetilde{P}_\Lambda + \widetilde{P}_{-\Lambda}}{2},\tag{3}$$

$$\prod_{m=-\Lambda}^{\Lambda} (L - mI) = 0, \quad (x_\pm)^{2\Lambda+1} = 0.\tag{4}$$

Here $\widetilde{P}_m$ is the projection over the 1-dim subspace spanned by ψ_m, and k is a sufficiently large function of Λ, for example $k = k(\Lambda) = \Lambda^2(\Lambda+1)^2$.

$O(3)$-equivariant fuzzy sphere - The essentials

We previously built a $O(3)$-equivariant fuzzy sphere, formed by a sequence $\{\mathcal{A}_{\Lambda,3}\}_{\Lambda\in\mathbb{N}}$ of unitary irreducible representations $(\pi_{\Lambda,3},\mathcal{H}_{\Lambda,3})$ of $Uso(4)$ and the corresponding representation spaces were denoted by

$$\mathcal{H}_{\Lambda,3} := span\left\{\psi_l^m | l\in\mathbb{N}_0, m\in\mathbb{Z}, l\le\Lambda, |m|\le l\right\}, \quad \text{where} \quad \Lambda\in\mathbb{N}.$$

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The angular momentum operators $\{L_a\}$ and the coordinate operators $\{x_a\}$ (here $a=0,+,-$) are obtained from the corresponding ones $\{L_i\}_{i=1}^3$ and $\{x_i\}_{i=1}^3$ as follows:

$$L_\pm := \frac{L_1 \pm iL_2}{\sqrt{2}}, \qquad L_0 := L_3, \qquad x_\pm := \frac{x_1 \pm ix_2}{\sqrt{2}}, \qquad x_0 := x_3.$$

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Furthermore, their action is

$$L_0\psi_l^m = m\,\psi_l^m, \quad L_\pm\psi_l^m = \frac{\sqrt{(l\mp m)(l\pm m+1)}}{\sqrt{2}}\psi_l^{m\pm1}, \qquad (5)$$

$$x_a \psi_l^m = \begin{cases} c_l A_l^{a,m} \psi_{l-1}^{m+a} + c_{l+1} B_l^{a,m} \psi_{l+1}^{m+a} & \text{if } l < \Lambda, \\ c_\Lambda A_\Lambda^{a,m} \psi_{\Lambda-1}^{m+a} & \text{if } l = \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

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where $A_l^{a,m} = B_{l-1}^{-a,m-a}$ and

$$A_l^{0,m} = \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}},\, A_l^{\pm,m} = \frac{\pm 1}{\sqrt{2}}\sqrt{\frac{(l\mp m)(l\mp m-1)}{(2l-1)(2l+1)}}, \qquad (7)$$

$$c_l := \sqrt{1+\frac{l^2}{k}} \qquad 1 \le l \le \Lambda, \qquad c_0 = c_{\Lambda+1} = 0, \qquad (8)$$

$$\text{with} \quad k = k(\Lambda) = \Lambda^2(\Lambda+1)^2$$

Moreover, we introduced the operator $\boldsymbol{x}^2 := x_i x_i = x_a x_{-a}$, which represents the square distance from the origin, and we showed that

$$\boldsymbol{x}^2 = 1 + \frac{L^2+1}{k} - \left[1 + \frac{(\Lambda+1)^2}{k}\right] \frac{\Lambda+1}{2\Lambda+1} \widetilde{P}_\Lambda \tag{9}$$

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In conclusion, we proved that

$$x_i^\dagger = x_i, \quad L_i^\dagger = L_i, \quad [L_i, x_j] = i\varepsilon^{ijh} x_h, \quad [L_i, L_j] = i\varepsilon^{ijh} L_h, \quad x_i L_i = 0$$

$$[x_i, x_j] = i\varepsilon^{ijh} \left(-\frac{I}{k} + K \widetilde{P}_\Lambda \right) L_h \quad i, j, h \in \{1, 2, 3\},$$

$$\prod_{l=0}^{\Lambda} \left[L^2 - l(l+1)I\right] = 0, \qquad \prod_{m=-l}^{l} (L_3 - mI) \widetilde{P}_l = 0, \ (x_\pm)^{2\Lambda+1} = 0.$$

Coherent states - Preliminaries

The notion of coherent states (CS) has arisen with the problem of saturating the quantum uncertainty relation (UR) on $\mathbb{R}^D$; on other manifolds M nontrivial problem! On $M = \mathbb{R}^D$ CS make up an overcomplete set in $\mathcal{H} := \mathcal{L}^2\left(\mathbb{R}^D\right)$ yielding a nice resolution of the identity. These properties are usually taken as minimal requirements for defining CS on other M. A set of coherent states $\{\phi_l\}$ is a particular set of vectors of a Hilbert space $\mathcal{H}$, l is an element of an appropriate label (and topological) space Ω s.t.:

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- **COMPLETENESS** (RESOLUTION ON UNITY): $\exists dl$ s.t.

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- (WEAKER) **COMPLETENESS** (TOTAL SET OF VECTORS):

$$\overline{span\left\{\phi_l : l \in \Omega\right\}} = \mathcal{H}.$$

Coherent states - Perelomov

In particular A. Perelomov (cf. his book) chooses $\Omega = G$: the system of CS $\{T, \phi_0\}$ is the set $\{\phi_g = T(g)\phi_0\}_{g \in \frac{G}{H}}$, where G is an arbitrary Lie group, H the isotropy subgroup of $\phi_0, T(g)$ is an unitary irreducible representation acting on a Hilbert space $\mathcal{H}$ and $\phi_0 \in \mathcal{H}$.

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The system of states which is as close as possible to the classical states is obtained, according to Perelomov, once one chooses ϕ_0 as the state for which the isotropy subalgebra is maximal; in the cases of our interest those states correspond to the ones which minimize the dispersion of the quadratic Casimir C_2.

Heisenberg UR - analog on S^1

Let $L = -i(x_1\partial_2 - x_2\partial_1)$ a.m. operator on $\mathbb{R}^2$. $[L, x_1] = ix_2$, $[L, x_2] = -ix_1$ implies the UR

$$\Delta L^2(\Delta x_1)^2 \geq \frac{1}{4}\langle x_2\rangle^2, \Delta L^2(\Delta x_2)^2 \geq \frac{1}{4}\langle x_1\rangle^2 \Rightarrow \Delta L^2(\Delta \boldsymbol{x})^2 \geq \frac{1}{4}\langle \boldsymbol{x}\rangle^2;$$

valid also on $\mathcal{H} = \mathcal{L}^2(S^1)$, but under $\boldsymbol{x}^2 \equiv x_1^2 + x_2^2 = 1$; 3rd ineq is a lower bound for $\Delta L\,|\Delta \boldsymbol{x}|$ in phase space.

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$$I = \sum_n P_n \overset{*}{=} \int_{G/H} P_x\, d\mu(x), \qquad P_n := \psi_n\langle\psi_n, \cdot\rangle.$$

$$G := \{(x_+)^n e^{i(aL+b)} \,|\, (a,b,n) \in \mathbb{R}^2 \times \mathbb{Z}\} \simeq U(1) \times U(1) \ltimes \mathbb{Z}$$

$H = \{e^{i(aL+b)}\} \simeq [U(1)]^2$ is the isotropy subgroup of ψ_0, and $G/H = \{(x_+)^n \,|\, n \in \mathbb{Z}\}$, hence $*$ integrating over G/H amounts to summing over $n \in \mathbb{Z}$,

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UR on S^2

$[L_i, L_j] = i\varepsilon^{ijk} L_k$, $[L_i, x^j] = i\varepsilon^{ijk} x^k$valid on $\mathcal{L}^2(\mathbb{R}^3)$, and $\mathcal{L}^2(S^2)$ implies the UR

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Coming back to Perelomov CS, if we take the irrep (π_Λ, V_Λ) of $Uso(3)$, characterized by

$$\boldsymbol{L}^2 = L_1^2 + L_2^2 + L_3^2 = \Lambda(\Lambda + 1),$$

it's easy to see that the dispersion $(\Delta \mathbf{L})^2 := \sum_i \Delta L_i^2$ of $\mathbf{L}$ is minimal for vectors $Y_\Lambda^{\pm\Lambda}$ and its explicit value is $(\Delta \mathbf{L})^2_{min} = \Lambda$.

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Proposition

The following UR holds on $\mathcal{L}^2(S^2)$, and is saturated by the spin (and Perelomov) CS belonging to each V_Λ, $\Lambda \in \mathbb{N}_0$.

$$\Delta\boldsymbol{L}^2 \geq |\langle\boldsymbol{L}\rangle| \qquad \Leftrightarrow \qquad \boldsymbol{L}^2 \geq |\langle\boldsymbol{L}\rangle|\,(|\langle\boldsymbol{L}\rangle|+1)\,.$$

Applying Perelomov construction one finds the resolution of the identity

$$I = c \sum_{l=0}^{\infty} \int_{SO(3)} d\mu(g) P_{l,g}, \quad P_{l,g} = \phi_{l,g} \langle \phi_{l,g}, \cdot \rangle, \quad \phi_{l,g} := T(g) Y_l^l.$$

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We can use these arguments for the Madore fuzzy sphere, because of the isomorphism

$$x_i = \frac{2L_i}{\sqrt{n^2-1}}, \quad i = 1, 2, 3$$

between the algebra of observables M_n and a suitable irreducible representation (π_Λ, V_Λ) of $Uso(3)$, having dimension $n = 2\Lambda + 1$.

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between the algebra of observables M_n and a suitable irreducible representation (π_Λ, V_Λ) of $Uso(3)$, having dimension $n = 2\Lambda + 1$.
Also in this case the dispersion of $\boldsymbol{x}^2 := x_i x_i \equiv 1$ is minimal on the states $Y_\Lambda^{\pm\Lambda}$ and it is

$$(\Delta\mathbf{x})^2_{\min} = \frac{2(n-1)}{n^2-1} = \frac{1}{\Lambda+1}. \tag{10}$$

Coherent states - Our targets

Our first target is give a meaningful definition of spatial dispersion $(\Delta \boldsymbol{x})^2$ on our fuzzy spaces, which will be a good measure of the localization of a state in configuration space $\mathbb{R}^D$; so we adopt the expectation value (variance)

$$(\Delta \boldsymbol{x})^2 := \left\langle (\boldsymbol{x} - \langle \boldsymbol{x} \rangle)^2 \right\rangle = \langle \boldsymbol{x}^2 \rangle - \langle \boldsymbol{x} \rangle^2 = \langle \boldsymbol{x}^2 \rangle - \sum_{i=1}^{D} \langle x_i \rangle^2 \quad (11)$$

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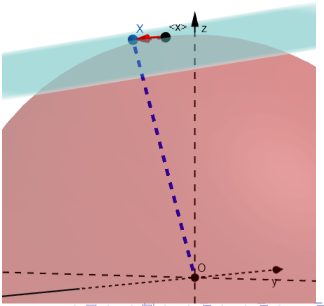
on the state; to motivate this choice we note that it is manifestly $O(D)$-invariant and that if the state is localized in a small region $\sigma_{\langle \boldsymbol{x} \rangle} \subset S^d$ around a point $\langle \boldsymbol{x} \rangle \in S^d$ then $(\Delta \boldsymbol{x})^2$ essentially reduces to the average square displacement in the tangent plane at $\langle \boldsymbol{x} \rangle$, as one wishes.



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According to this, an analysis of $\Sigma_{x_i}(\Lambda)$ is our second target.

The third target is determine the most *localized* states of our fuzzy spaces, i.e. the ones which minimize the spatial dispersion $(\Delta \boldsymbol{x})^2$, and (as we will see) this target is strictly linked to the previous one.

If $\widehat{\chi}$ is a state of our $O(2)$-equivariant fuzzy circle, then the construction of Perelomov with $G \equiv SO(3)$ (which is a larger group of the algebra automorphisms), $T \equiv \pi_{\Lambda,2}$ and $\mathcal{H} \equiv \mathcal{H}_{\Lambda,2}$ returns us a system of states which fulfills the **CONTINUITY** and **COMPLETENESS** properties;

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The analogous situation occurs for our $O(3)$-equivariant fuzzy sphere.
For this reason, we try to apply the construction of Perelomov to $\widehat{\chi}$ also with $G \equiv O(2)$ for the circle and $O(3)$ for the sphere. This is our fourth target

Coherent states - The x^i eigenvalue-problem

The covariance of the algebra under $O(D)$ transformations $\boldsymbol{x} \mapsto \boldsymbol{x}' = R\boldsymbol{x}$, $\boldsymbol{L} \mapsto \boldsymbol{L}' = R\boldsymbol{L}$ implies that the spectrum $\Sigma_{x_i}(\Lambda)$ of any coordinate operator x_i of our fuzzy spaces is the same, so we can focus our attention on the spectra of x_1 and x_3 when $D = 2$ and $D = 3$, respectively.

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For convenience, we study the eigenvalue equations

$$x_1\psi = \alpha_1\psi \text{ when } D = 2 \quad \text{and} \quad x_3\psi = \alpha_2\psi \text{ when } D = 3. \qquad (12)$$

We don't need to make two different discussions, one for $D = 2$ and one for $D = 3$, for this reason now we start analyzing the three-dimensional problem.

First of all, we defined $x_0 := x_3$ and $L_0 := L_3$, but it's also true that the problem of maximizing $\langle x_0 \rangle$ is equivalent to the one of finding the maximal eigenvalue of x_0; in conclusion, because of $[x_0, L_0] = 0$, we can simoultaneously diagonalize x_0 and L_0.

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However, we have to impose this system

$$\begin{cases} L_0 \boldsymbol{\chi}_\alpha^\beta = \beta \boldsymbol{\chi}_\alpha^\beta \\ x_0 \boldsymbol{\chi}_\alpha^\beta = \alpha \boldsymbol{\chi}_\alpha^\beta \end{cases}, \tag{13}$$

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then, using (6) we can easily conclude that

$$\beta = m \in \{-\Lambda, \cdots, \Lambda\} \quad \text{and} \quad \boldsymbol{\chi}_\alpha^m = \sum_{l=|m|}^{\Lambda} \chi_l^m \psi_l^m. \tag{14}$$

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Proposition

If $\widetilde{\alpha}$ is an eigenvalue of x_0, then also $-\widetilde{\alpha}$ must be an eigenvalue of x_0.

We proved also that

Theorem

Let α_0 be the maximal eigenvalue of x_0 and χ_0 be the corresponding eigenvector. It is such that

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$$L_3 = (\underline{\boldsymbol{L}})_3 = \left(\underline{\boldsymbol{r}} \times \underline{\boldsymbol{p}}\right)_3 = 0,$$

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Theorem

The maximal eigenvalue $\alpha_0\left(\Lambda\right)$ of x_0 fulfills

$$\lim_{\Lambda\to+\infty} \alpha_0\left(\Lambda\right) = 1.$$

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$\Sigma_{x_0}(\Lambda)$ *and* $\Sigma_{x_0}(\Lambda + 1)$ *interlace.*

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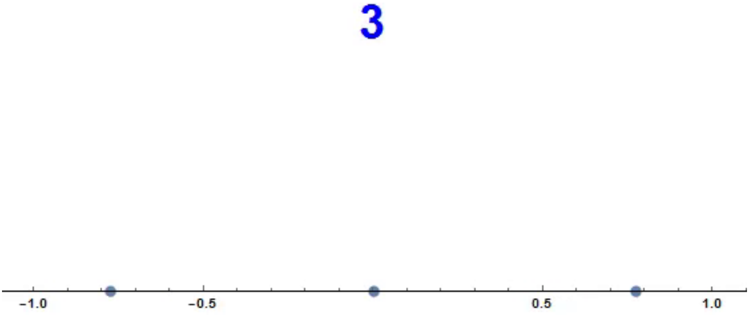

Figure 1: The spectrum $\Sigma_{x_0}(\Lambda)$ when $\Lambda = 2, \cdots, 100$.

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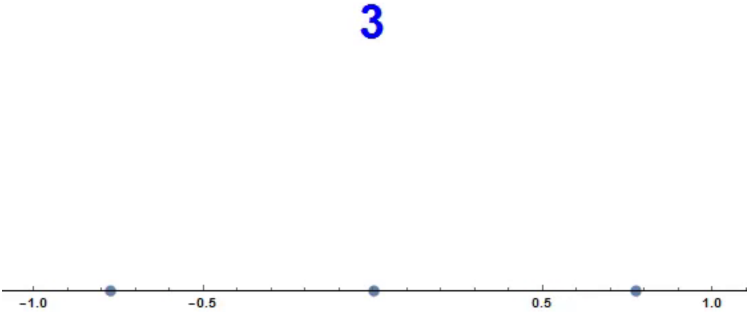

Figure 1: The spectrum $\Sigma_{x_0}(\Lambda)$ when $\Lambda = 2, \cdots, 100$.

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Theorem

The spectrum $\Sigma_{x_0}(\Lambda)$ of x_0 becomes dense in $[-1,1]$ as $\Lambda \to +\infty$.

Coherent states - Most localized states

Now we want to solve the problem of getting the most *localized* states of our fuzzy spaces, as seen previously we want to minimize $(\Delta \boldsymbol{x})^2$; but the $O(D)$-covariance implies that $(\Delta \boldsymbol{x})^2_\psi = (\Delta R\boldsymbol{x})^2_\psi$ for every state $\psi \in \mathcal{H}_{\Lambda,D}$ and $O(D)$- transformation R;

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On the other hand, in both dimensions $\boldsymbol{x}^2 = 1$ up to $\frac{1}{\Lambda^2}$, so the problem of minimizing $(16)_1$ is strictly linked to the one of maximizing $\langle x_1 \rangle$ for $D = 2$, as for $(16)_2$ and $\langle x_3 \rangle$ for $D = 3$.

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$$\boldsymbol{\chi} = \widetilde{\boldsymbol{\chi}} := \sum_{l=0}^{\Lambda} \widetilde{\chi}^l \boldsymbol{\psi}_l^0, \quad \text{with} \quad \widetilde{\chi}^l = \frac{\sin\left[\frac{(l+1)\pi}{\Lambda+2}\right]}{\sqrt{\frac{\Lambda+2}{2}}} \quad \text{if } 0 \le l \le \Lambda,$$

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According to the last inequality and (9) we can infer

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$$(\Delta \boldsymbol{x})^2_{\widetilde{\widetilde{\chi}}} = \left[\frac{\pi}{2(\Lambda + 1)}\right]^2 + O\left(\frac{1}{\Lambda^3}\right).$$

Comparison with Perelomov & Madore

More precisely, we will compare the spatial dispersion $(\Delta \boldsymbol{x})^2$ of the Madore fuzzy sphere with the our $(\Delta \boldsymbol{x})^2$ when the representation spaces are

$$V_\Lambda := span\{Y_\Lambda^m : -\Lambda \leq m \leq \Lambda\}$$

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So, it's obvious that (definitively)

$$(\Delta \boldsymbol{x})^2_{min} \leq (\Delta \boldsymbol{x})^2_{\widetilde{\chi}} < (\Delta \boldsymbol{x})^2_{min}.$$

Fourth target

If $D = 2$ and we adopt $T = \pi_{\Lambda,2}$ and as G not $SU(2)$ but its subgroup $G = U(1)$; hence $\mathcal{H}_{\Lambda,2}$ carries a *reducible* representation of G, and **COMPLETENESS** (RESOLUTION OF UNITY) is not automatic.

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One can prove that if $\phi = \sum_{m=-\Lambda}^{\Lambda} \phi_m \psi_m$, $\|\phi\| = 1$ and

$$\phi_\alpha := e^{i\alpha L}\phi = \sum_{m=-\Lambda}^{\Lambda} e^{i\alpha m}\phi_m\psi_m, \quad P_\alpha := \phi_\alpha\langle\phi_\alpha, \cdot\rangle = |\phi_\alpha\rangle\langle\phi_\alpha|$$

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$(\boldsymbol{\phi}_0 \equiv \boldsymbol{\phi})$;then defining $B := \int_0^{2\pi} d\alpha\, P_\alpha$ it turns out that $B \propto I$ if and only if $|\phi_m|$ is independent on n, which implies $|\phi_m| = 1/(2\Lambda + 1)$, but in this case

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this goes to zero as $\Lambda \to \infty$, but more slowly than the spatial dispersion of $\widetilde{\widetilde{\chi}}$ and we proved that the same applies for $D = 3$.

In $D = 3$ we choose G not $SO(4)$ but its subgroup $G = SO(3)$, and $T = \boldsymbol{\pi}_\Lambda$. The $(\mathcal{H}_\Lambda, \boldsymbol{\pi}_\Lambda)$ is a *reducible* representation of G, more precisely the direct sum of the irreducible representations (V_l, π_l), $l = 0, ..., \Lambda$, therefore completeness and resolution of unity are not automatic.

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$$\boldsymbol{\phi}_g:=\boldsymbol{\pi}_\Lambda(g)\boldsymbol{\phi}, \qquad\qquad P_g:=\boldsymbol{\phi}_g\langle\boldsymbol{\phi}_g,\cdot\rangle \tag{18}$$

$(\boldsymbol{\phi}_l\equiv\boldsymbol{\phi})$. The system $A:=\{\boldsymbol{\phi}_g\}_{g\in G}$ is complete provided $\phi_l\neq 0$ for all l (then it is also overcomplete). Defining $B:=\int_{SO(3)}d\mu(g)\,P_g$ one finds that B is proportional to the identity only if $|\phi_l|^2$ is independent of l and therefore (since $\|\boldsymbol{\phi}\|=1$) if $|\phi_l|^2=1/(\Lambda+1)$. Setting $\phi_l=e^{i\beta_l}/\sqrt{\Lambda+1}$ $(\beta_l\in\mathbb{R})$ we find the following resolutions of the identity, parametrized by $\beta\in(\mathbb{R}/2\pi\mathbb{Z})^{\Lambda+1}$:

$$I=\frac{\Lambda+1}{2\pi^3}\int_{SO(3)}d\mu(g)\,P_g^\beta,\, P_g^\beta:=\boldsymbol{\psi}_g^\beta\langle\boldsymbol{\psi}_g^\beta,\cdot\rangle,\,\boldsymbol{\psi}_g^\beta:=\sum_{m=-\Lambda}^{\Lambda}\frac{e^{i\beta_l}}{\sqrt{\Lambda+1}}\boldsymbol{\pi}_\Lambda(g)\boldsymbol{\psi}_l^l.$$

Appendix

Another possibility is to minimize the variance of $\pi_{\Lambda,2}\left[\boldsymbol{L}^2\right]$, but one can easily show that in this case the "coherent states" (the ones minimizing that variance) are ψ_{Λ} and $\psi_{-\Lambda}$, which are not meaningful in the high energy limit $\Lambda \to +\infty$.

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Incidentally, some authors consider also two definitions of sets of optimally localized states on the spin sphere alternative to the one adopted by Perelomov: the set of "intelligent states", that saturate the uncertainty relation $\Delta L_1 \, \Delta L_2 \geq |\langle L_3 \rangle|/2$, and the set of "minimum uncertainty states", for which $\Delta L_1 \, \Delta L_2$ has a local minimum (note that then in general $\Delta L_1 \, \Delta L_3$, $\Delta L_2 \, \Delta L_3$ are not minimized). But neither one is invariant under arbitrary rotation, in contrast with the definition of Perelomov and of the present work; one can easily show that these states are "fewer" than the points of S^2, i.e cannot be put in one-to-one correspondence with the points of S^2, but just of a finite number of lines on S^2.