Hidden geometry of the Standard Model

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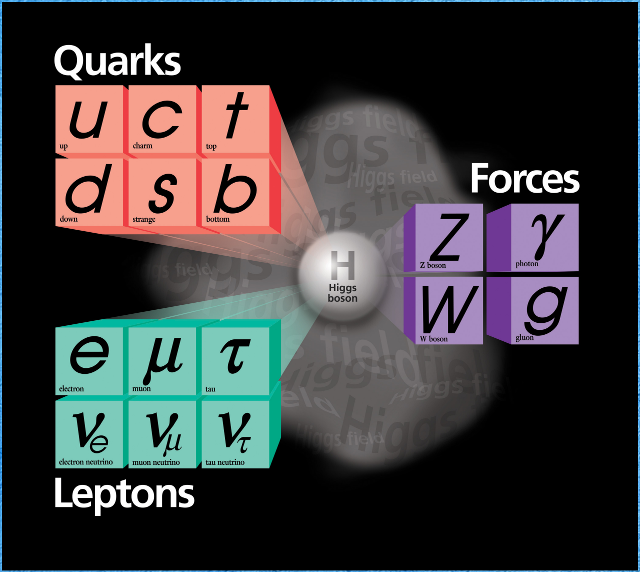
Physics and Geometry



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2. spin geometry is seen through the existence of the Dirac operator constructed from torsion-free spin connection
3. spinors are sections of a complex vector bundle with a representation of the Clifford algebra
4. the structure of the spinor bundle and the Dirac operator encodes the chirality and Lorentzian structure
5. the spinor bundle of fermions is an irreducible representation of the chiral Clifford algebra (and that can be encoded in saying that tbe bundle establishes the Morita equivalence between functions and Clifford algebra).

Spectral triples

Definition: a geometry (algebraic way)

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions.

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THEOREM [CONNES]

If you have a spin manifold then the Dirac operator acting on the sections of the spinor bundle gives you geometry in the above sense.

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Theorem [Connes]

If you have a spin manifold then the Dirac operator acting on the sections of the spinor bundle gives you geometry in the above sense.

Remark

Another natural triple is $(C^\infty(M), L^2(\Omega(M)), d + d^*)$.

The structure of the Standard Model

Testing the Euclidean SM

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3. the full geometry of SM is (almost) a product geometry
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Finite spectral triple of the SM

The "almost commutative" geometry is described by the Hilbert space

$$H_F = \mathbb{C}^{96} =: H_f \otimes \mathbb{C}^3,$$

$$\begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} \quad \begin{bmatrix} \bar{\nu}_R & \bar{e}_R & \bar{\nu}_L & \bar{e}_L \\ \bar{u}_R^1 & \bar{d}_R^1 & \bar{u}_L^1 & \bar{d}_L^1 \\ \bar{u}_R^2 & \bar{d}_R^2 & \bar{u}_L^2 & \bar{d}_L^2 \\ \bar{u}_R^3 & \bar{d}_R^3 & \bar{u}_L^3 & \bar{d}_L^3 \end{bmatrix}$$

The spectral triple of the SM

The algebra for g generations

$$A_F = (M_1 \oplus M_1 \oplus M_2)^{(4g)} \oplus (M_1 \oplus M_3)^{(4g)},$$

with first M_1 identified with each other:

$$(z \oplus w \oplus h)^{(4g)} \oplus (z \oplus m)^{(4g)}.$$

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THE DIRAC OPERATOR

$$\tilde{D}_F = \left(D_l \oplus D_q^{(3)} \right) \oplus 0^{(16g)} \,,$$

The Dirac operator and Hodge duality

The Dirac operator

$$D_l = \left(\begin{array}{cc|cc} 0 & 0 & \Upsilon_\nu & 0 \\ 0 & 0 & 0 & \Upsilon_e \\ \hline \Upsilon_\nu^* & 0 & 0 & 0 \\ 0 & \Upsilon_e^* & 0 & 0 \end{array}\right), \quad D_q = \left(\begin{array}{cc|cc} 0 & 0 & \Upsilon_u & 0 \\ 0 & 0 & 0 & \Upsilon_d \\ \hline \Upsilon_u^* & 0 & 0 & 0 \\ 0 & \Upsilon_{d*} & 0 & 0 \end{array}\right),$$

where where $D_l, D_q \in M_{4g}$ are positive mass matrices for leptons and quarks, respectively.

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Question

What is the algebra generated by $\tilde{D}_F$ and A_F and what is its commutant ? If the commutant of $Cl(A_F, D_F)$ is isomorphic to A_F then SM correponds to fundamental spinors. *BUT...*

The Hodge Duality

SM as geometry

If the algebra generated by $A_F, \tilde{D}_F$ is

$$M_{4g} \oplus M_{4g}^{3} \oplus M_{1}^{4g} \oplus M_{3}^{4g}$$

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Does it happen in reality?

Yes, if there is nontrivial mixing of the neutrino flavors and nontrivial mixing of the quarks and all fermion (bare) masses are nonzero and dfifferent from each other then the SM satisfies the Hodge duality.

ONE GENERATION

In this case the various Υ's are just complex numbers and so the D_l and D_q matrices are just 4×4 complex matrices acting on the spaces of leptons and quarks, where Υ_e is the electron mass and Υ_ν is the neutrino mass (and similarly for D_q).

Since $\mathbb{C}l_D(A)$ contains $M_1^{(4)} \oplus M_3^{(4)}$, the commutant $\mathbb{C}l_D(A)'$ of $\mathbb{C}l_D(A)$ must contain $M_4 \oplus M_4^{(3)}$, and, if the Hodge duality is satisfied, so must $\mathbb{C}l_D(A)$. However, $\mathbb{C}l_D(A)$ contains $M_1^{(4)} \oplus M_3^{(4)}$ and two algebras generated respectively by $M_1 \oplus M_1 \oplus M_2$ and D_l, and $(M_1 \oplus M_1 \oplus M_2)^{(3)}$ and $D_q^{(3)}$. Thus the only possibility that the Hodge condition holds is when these two algebras are M_4 and $M_4^{(3)}$, respectively. This happens only when independently all $z \oplus w \oplus h \in M_1 \oplus M_1 \oplus M_2$ and D_l, as well as $z \oplus w \oplus h$ and D_q, each generate M_4. It is easy to notice that sufficient and necessary condition for this is that all four masses Υ's are different from zero.

Three generations

Leptons

We start with leptons and check whether the algebra generated by A_l, D_l is a full matrix algebra. A general matrix that commutes with A_l has a form $P_1 \oplus P_2 \oplus \tilde{P}_3$, where $P_1, P_2 \in M_g$ and $\tilde{P}_3 = 1 \otimes P_3 \in M_2 \otimes M_g$. If it commutes with D_l then:

$$P_1 \Upsilon_\nu = \Upsilon_\nu P_3, \qquad P_2 \Upsilon_e = \Upsilon_e P_3,$$
$$P_3 \Upsilon_\nu^* = \Upsilon_\nu^* P_1, \qquad P_3 \Upsilon_e^* = \Upsilon_e^* P_1.$$

From these equations we immediately infer that P_1 and P_3 must commute with $\Upsilon_\nu \Upsilon_\nu^*$ (note that since both Υ matrices are unitarily similar to diagonal matrix then they are normal) whereas P_2 and P_3 must commute with $\Upsilon_e \Upsilon_e^*$.

Three generations

...and quarks

Therefore, only if both Υ_ν and Υ_e are invertible and the pair $\Upsilon_\nu \Upsilon_\nu^*$ and $\Upsilon_e \Upsilon_e^*$ generate the full matrix algebra M_g it follows consequently that P_1 and P_2 must be equal to P_3, and be proportional to the identity. Similar arguments will also hold for the quarks: it suffices (and is necessary) that the two matrices $\Upsilon_u \Upsilon_u^*$ and $\Upsilon_d \Upsilon_d^*$ generate the full matrix algebra M_g and that they are invertible to assure that the algebra generated by A_q, D_q is a full matrix algebra.

Masses

When two hermitian matrices, A, B in $M_3(\mathbb{C})$ generate a full matrix algebra ?
Burnside (1905): they do not share a common eigenvector (the theorem states that there is no common invariant subspace but since the matrices are hermitian if there exists an invariant subspace its complement is also invariant and hence there would necessarily exist an invariant subspace of dimension 1).

If, we assume that all eigenvalues of A are different from each other then we only need to check the matrix elements of U in the chosen basis, in which A is diagonal. If no matrix element of U is of modulus 1 (while at the same time other matrix elements in the same row and in the same column are 0) then U does not map one of the basis vectors to another one. Equivalently, one can reformulate the condition in the following way: no permutation of the basis leads to the block diagonal matrix of U with rank of the largest block strictly less than 3.

The physical parameters

$$U = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix},$$

where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$. The following Jarlskog invariant:

$$J_{\nu CP}^{max} = \frac{1}{8}\cos(\theta_{13})\sin(2\theta_{13})\sin(2\theta_{23})\sin(2\theta_{12}),$$

provides for the neutrino mixing range (with 1σ error):

$$J_{\nu CP}^{max} = 0.0329 \pm 0.0007,$$

that proves indeed that not only all the angles are non-vanishing but all neutrino masses are different from each other.

Here the usual convention in physics is different with "up" sector diagonal and "down" sector mixed. The bare up and down quark massed are different from each other within the errors so the only thing to check is the mixing matrix U. Using again the same type of parametrization of the matrix by three angles and the phases, we can just look at the experimental value of the Jarlskog invariant J_{qCP}^{max}, measured with 1σ:

$$J_{qCP}^{max} = \left(3.04\frac{+0.21}{-0.20}\right) 10^{-5},$$

which is sufficient to ascertain that all angles are indeed nonzero and that implies the partial condition to Hodge duality.

Note that unlike in the leptonic case the angles are very small, which means that the matrix U is very close to the diagonal unit matrix. Nevertheless within the experimental errors we see that the quark mixing is also maximal and the Clifford algebra generated in the quark sector is the full matrix algebra M_{4g} as well.

The Lorentzian spectral triples

The definition

A real pseudo-Riemannian spectral triple of signature (p, q) is a system $(\mathcal{A}, \pi, \mathcal{H}, D, J, \gamma, \beta)$ where $\mathcal{A}$ is an involutive unital algebra, π its faithful $*$-representation on an Hilbert space $\mathcal{H}$. First, for even $p+q$ there exists a $\mathbb{Z}_2$-grading $\gamma^\dagger = \gamma, \gamma^2 = 1$ commuting with the representation of $\mathcal{A}$, J is an antilinear isometry and for all $a, b \in \mathcal{A}$ we have $[J\pi(a^*)J^{-1}, \pi(b)] = 0$. Furthermore, there exists an additional grading $\beta = \beta^\dagger, \beta^2 = 1$ also commuting with the representation of $\mathcal{A}$, which defines the Krein structure on the Hilbert space. The latter is an indefinite bilinear form defined as $(\phi, \psi)_\beta = (\phi, \beta\psi)$, where $(\cdot, \cdot)$ is the usual positive definite scalar product on the Hilbert space. As a last requirement, we postulate the existence of a (possibly unbounded) densely defined operator D, which is β-self-adjoint, i.e. $D^\dagger = (-1)^p \beta D \beta$ and such that $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$, is odd with respect to γ-grading: $D\gamma = -\gamma D$.

Finite Lorentzian triples

The Euclidean part of FST

Define $D_+ = \frac{1}{2}(D + D^\dagger)$ and $D_- = \frac{i}{2}(D - D^\dagger)$. Both $D_\pm$ are by definition self-adjoint and we obtain a pair of Riemannian real spectral triples, $(\mathcal{A}, \pi, \mathcal{H}, D_\pm, J, \gamma)$.

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3. If for an even spectral triple grading is scalar on at least one subspace H_{ij} then there is no pseudo-Riemannian triple with p odd.
4. The Euclidean part of ST is a finite spectral triple *BUT* with an additional symmetry β of the Dirac operator.

The lepton number symmetry from Lorentzian structure

The pseudo-Riemannian structure

The finite spectral triple of the Standard Model is consistent with it being the Euclidean part of the pseudo-Riemannian triple with β being:

$$\beta = \pi_F(1,1,-1) J_F \pi_F(1,1,-1) J_F^{-1},$$

where $\lambda, h, m \in M_1 \oplus M_2 \oplus M_3$.

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Is this unique ?

Yes, it is the only possible 0-cycle (β) for a real spectral triple over the Standard Model that can be interpreted shadow of a pseudo-Riemannian structure, which additionally allows Hodge duality is the one with $\beta = \pi(1, 1, -1) J \pi(1, 1, -1) J^{-1}$.
This results in the symmetry that physically can be interpreted as **lepton number conservation.**

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Literature

A. Bochniak, A.Sitarz, Finite Pseudo-Riemannian Spectral Triples and The Standard Model Phys. Rev. D 97, 115029 (2018)

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