Poisson-Lie T-duality and generalized geometry

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Joint work with Fridrich Valach

What is Poisson-Lie T-duality?

[Klimčík, Š. 1995]

T-duality

Two different spacetimes $M_{1,2}$ can be equivalent from the string theory perspective
Requires an action of $U(1)$ (or of a torus) on M_1 by isometries

Poisson-Lie T-duality

- A non-Abelian generalization (symmetry is hidden, no Killing vector fields)
- M_1 and M_2 give isomorphic Hamiltonian systems (up to finitely many degrees of freedom)

Courant algebroids, or "generalized geometry"

[Liu, Weinstein, Xu 1997]

Courant algebroid: vector bundle $E \to M$, symmetric pairing $\langle \, , \, \rangle$ anchor map $\rho : E \to TM$, bracket $[\, , \,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ such that $(\forall s, t, u \in \Gamma(E))$

$$\begin{aligned} [s, [t, u]] &= [[s, t], u] + [t, [s, u]] \\ \rho(s)\langle t, u \rangle &= \langle [s, t], u \rangle + \langle t, [s, u] \rangle \\ \langle s, [t, t] \rangle &= \langle [s, t], t \rangle . \end{aligned}$$

Examples

- Lie algebras with invariant symmetric pairing ($M =$ point)
- **exact CAs** (classified by $H^3(M, \mathbb{R})$)

$$0 \to T^*M \xrightarrow{\rho^t} E \xrightarrow{\rho} TM \to 0$$

2d σ-models and generalized metrics

2d σ-model

Ingredients: (M, g, H): g a Riemannian metric, $H \in \Omega^3(M)_{\text{closed}}$
Σ a surface with a Lorentzian metric

$$S(f) = \int_\Sigma g(\partial_+ f, \partial_- f) + \int_Y f^* H \qquad (f : \Sigma \to M,\ \partial Y = \Sigma)$$

Generalized metric in a CA $E \to M$:
a subbundle $V_+ \subset E$, maximally positive-definite w.r.t. $\langle , \rangle$

Observation

(M, g, H) is the same as a generalized metric
in an *exact CA* $E \to M$

CAs and Hamiltonian systems

- A CA $E \to M \rightsquigarrow$ a symplectic manifold $L_{CA}E$
- A generalized metric $V_+ \subset E \rightsquigarrow$ a function $\mathcal{H}_{V_+}$ on $L_{CA}E$

If E is exact, we get the σ-model:

$L_{CA}E \cong T^*(LM)$ (the phase space of the σ-model)
$\mathcal{H}_{V_+}$ = the Hamiltonian of the σ-model

Poisson-Lie T-duality

Backgrounds (M, g, H) of Poisson-Lie type

- a Courant algebroid $\tilde{E} \to \tilde{M}$ (not exact), $\tilde{V}_+ \subset \tilde{E}$
- a surjective submersion $f : M \to \tilde{M}$
- a compatible *exact* CA structure on $E := f^*\tilde{E} \to M$ **(not unique !)**
- pulled-back generalized metric: $V_+ := f^*\tilde{V}_+ \subset E$, gives rise to (g, H) on M

PL T-duality

If (M_1, g_1, H_1) and (M_2, g_2, H_2) are obtained by pulling back the same gen. metric $\tilde{V}_+ \subset \tilde{E}$ then the corresponding 2-dim σ-models are (almost) isomorphic as Hamiltonian systems

...because they are (almost) isomorphic to $(L_{CA}\tilde{E}, \mathcal{H}_{\tilde{V}_+})$

How to construct CA pullbacks

No spectators (i.e. $\tilde{M}$ = point, $\tilde{E} = \mathfrak{d}$ a Lie algebra, $\tilde{V}_+ \subset \mathfrak{d}$)

- $\mathfrak{g} \subset \mathfrak{d}$ a Lagrangian Lie subalgebra ($\mathfrak{g}^\perp = \mathfrak{g}$)
- $M = D/G$, $E = \mathfrak{d} \times M$, the anchor given by the action of $\mathfrak{d}$
- (g, H) given by the gen. metric $\tilde{V}_+ \times M \subset \mathfrak{d} \times M$

Different $\mathfrak{g}$'s give PL-equivalent (M, g, H)'s

General $\tilde{M}$ (= spectators)

- A principal D-bundle $P \to \tilde{M}$
- Vanishing 1st Pontryagin class:
  $\langle F, F\rangle/2 = dC$ ($C \in \Omega^3(\tilde{M})$) gives a transitive CA $\tilde{E} \to \tilde{M}$
- $M = P/G$

A better description: A multiplicative gerbe over D trivial on G, acting on a gerbe on P

“Quantum questions”

σ-models:

is PL T-duality compatible with the renormalization group flow?

$$\frac{d}{dt}\, g = \mathrm{Ric}$$

– looking for suitable Ric of generalized metrics

string theory:

other massless fields besides (g, H): dilaton, RR-fields, gauge fields. Do they make sense for arbitrary CAs? Is PL T-duality compatible with SUGRA equations?

Generalized Ricci flow

There is a (almost) natural flow of generalized metrics:

Generalized Ricci flow (of a generalized metric)

$$\tfrac{d}{dt} V_+ = T_{V_+} : V_+ \to V_- \qquad \langle T_{V_+} u, v\rangle = \mathrm{GRic}_{V_+,\mathrm{div}}(u,v)$$

$$\mathrm{GRic}_{V_+,\mathrm{div}}(u,v) := \mathrm{div}[v,u]_+ - v\cdot \mathrm{div}\, u - \mathrm{Tr}_{V_+}[[\,\cdot\,,v]_-,u]_+$$

Here $\mathrm{div} : \Gamma(E) \to C^\infty(M)$ is such that $\mathrm{div}(fu) = f\,\mathrm{div}\,u + \rho(u)f$, e.g. $\mathrm{div}\,u := \mu^{-1}\mathcal{L}_{\rho(u)}\mu$ for a density μ [Alekseev,Xu 2001], [Garcia-Fernandes 2016].

Different choices of div give the same flows up to (inner) automorphisms of the CA.

Other definitions of GRic: [Coimbra, Strickland-Constable, Waldram 2011], [Garcia-Fernandez 2014], [Jurčo, Vysoký 2016] (using auxiliary data)

PL T-duality is compatible with the renorm. group flow

- If E is exact, the GRicci flow is the renormalization group flow (Ricci flow) of (g, H)
- GRic is compatible with CA pullbacks
- Hence, Poisson-Lie T-duality is compatible with the renormalization group flow

String effective action

Generating functional (string effective action)

There is a natural Laplacian Δ_{V_+} acting on half-densities. The GRicci flow is the gradient flow of

$$S(V_+, \sigma) := -\frac{1}{2} \int_M \sigma \, \Delta_{V_+} \sigma$$

(for a fixed σ).

For exact CA, with $\sigma = e^{-\phi} \mu_g^{1/2}$ ($V_+ \subset E$ corresponds to (g, H))

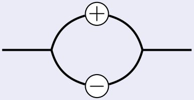
$$S(V_+, \sigma) = \int_M \left(\tfrac{1}{4} R - \tfrac{1}{8} H^2 + \|d\phi\|_g^2 \right) e^{-2\phi} \mu_g$$

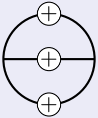
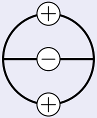
is the (bosonic) string effective action

For (suitable) transitive CAs it is the type I/heterotic SUGRA action [Garcia-Fernandez 2014]

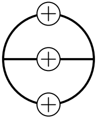
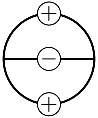
Who is who in the case of a Lie algebra

$$E = \mathfrak{d}, \quad V_+ \subset \mathfrak{d}$$

$$\mathrm{GRic}_{V_+}(u, v) = -\,\mathrm{Tr}_{V_+}[[\,\cdot\,, v]_-, u]_+ = -\; u$$  v

$$\Delta_{V_+} = \frac{1}{6}$$  $+ \frac{1}{2}$ 

In a general CA:

$$\Delta_{V_+} = 4\mathcal{L}_{\rho(e_a)}\mathcal{L}_{\rho(e_a)} + \frac{1}{6}$$  $+ \frac{1}{2}$ 

e_a an ON basis of V_+

PL T-duality and string effective action

without RR fields

Generalized string background equations

EOM of S: $\Delta_{V_+}\sigma = 0$, $\mathrm{GRic}_{V_+,\sigma} = 0$ (exact CAs: bosonic string background equations; some transitive CAs: type I/heterotic)

PL T-duality setup with a dilaton

$\tilde{V}_+ \subset \tilde{E}$, a half-density $\tilde{\sigma}$, $\quad M \xrightarrow{f} \tilde{M}$, $E := f^*\tilde{E}$ a CA pullback *and* an invariant fibrewise half-density τ:

$$\mathcal{L}_{\rho(f^*u)}\tau = 0 \quad \forall u \in \Gamma(\tilde{E}) \qquad \left(\Rightarrow \ \Delta_{V_+}(\tau\, f^*\tilde{\sigma}) = \tau\, f^*\Delta_{\tilde{V}_+}\tilde{\sigma}\right)$$

Example: $\tilde{E} = \mathfrak{d}$ ($\tilde{M} = pt$), $M = D/G$: τ exists iff G is unimodular

PL T-duality for string background equations

$(\tilde{V}_+, \tilde{\sigma})$ satisfy the GSBE iff $(V_+ := f^*\tilde{V}_+,\ \sigma := \tau\, f^*\tilde{\sigma})$ do

Another approach: [Jurčo, Vysoký 2018]

Type II: RR fields and generating Dirac operators

Stolen from [Coimbra, Strickland-Constable, Waldram 2011] in the case of exact CAs

RR-field: an E-spinor half-density F which is V_+-self-dual and $DF = 0$ (D is the Dirac generating operator of [Alekseev, Xu 2001])

(Pseudo)Action: $S(V_+, \sigma, F) = -\frac{1}{2}\int \Big(\sigma \Delta_{V_+}\sigma - \tfrac{1}{8}(F, *_{V_+} F)\Big)$

PL T-duality for type II SUGRA:

$(\tilde{V}_+, \tilde{\sigma}, \tilde{F})$ is a solution of the EOM in $\tilde{E}$ iff
$(V_+ = f^*\tilde{V}_+, \sigma = \tau\, f^*\tilde{\sigma}, F = \tau\, f^*\tilde{F})$ is a solution in $E = f^*\tilde{E}$

(If no τ exists we get a solution of modified type II SUGRA of [Tseytlin, Wulff 2016], [Arutyunov, Frolov, Hoare, Roiban, Tseytlin 2016])

Back to the worldsheet perspective

AKSZ model [Alexandrov, Kontsevich, Schwarz, Zaboronsky 1997]

$(\mathcal{M}, \omega, D)$ a dg symplectic manifold, $\deg \omega = n$

Y a closed $n+1$-dim manifold
$\Rightarrow \mathrm{Maps}(T[1]Y, \mathcal{M})$ is dg symplectic, $\deg \omega' = -1$, $D' = \{S, \cdot\}$

S is the AKSZ action (an $n+1$-dim TFT)
(S satisfies the classical master equation $\{S, S\} = 0$;
critical points of S = dg maps $T[1]Y \to \mathcal{M}$)

example: Chern-Simons
$\mathcal{M} = \mathfrak{d}[1]$, $\omega = \langle , \rangle$, $n = 2$

$$S(A) = \int_Y \left(\frac{1}{2} \langle A, dA \rangle + \frac{1}{6} \langle [A, A], A \rangle \right)$$

$A \in \Omega(\Sigma, \mathfrak{d})[1] = \mathrm{Maps}(T[1]Y, \mathfrak{d}[1])$

AKSZ for manifolds with a boundary

requires a boundary condition:

a dg Lagrangian submanifold $\Lambda \subset \mathrm{Maps}(T[1]\partial Y, \mathcal{M})$

$\Rightarrow \mathrm{Maps}(T[1]Y, \mathcal{M})_\Lambda \subset \mathrm{Maps}(T[1]Y, \mathcal{M})$ is dg symplectic

Our setup: $n = 2$, $\mathcal{M}$ equivalent to a CA E

$\Sigma = \partial Y$ with a Lorentzian metric

A gen. metric $V_+ \subset E$ produces a $\Lambda \subset \mathrm{Maps}(T[1]\Sigma, \mathcal{M})$

Example: $E = \mathfrak{d}$ (Chern-Simons)

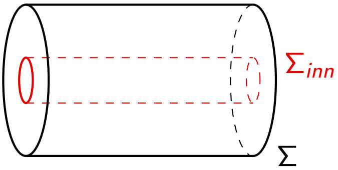
$$\Lambda = \{A \in \Omega^1(\Sigma, \mathfrak{d}) \mid *A = RA\} \oplus \Omega^2(\Sigma, \mathfrak{d}) \subset \Omega(\Sigma, \mathfrak{d})$$

$R : \mathfrak{d} \to \mathfrak{d}$ the reflection wrt. V_+

If E is exact this 3d AKSZ model is the 2d σ-model given by V_+

Open problems, if we take the AKSZ approach seriously

- Is its renormalization group flow equal to the GRicci flow?
- Is there a "string theory" (or SUGRA) that would explain the "dilaton" σ, the action $S(V_+, \sigma)$, the "RR field" F etc. also in the case of non-exact CAs?
- To what extent is the PL T-duality an exact equivalence? E.g. the σ-model with the target D/G is the $\mathfrak{d}$-Chern-Simons



$A|_\Sigma \in \Lambda$

$A|_{\Sigma_{inn}} \in \Omega(\Sigma_{inn}, \mathfrak{g})[1]$

- What is T-duality and what is its most general version? (A dg Lagrangian submanifold $L \subset \mathcal{M}_1 \times \overline{\mathcal{M}_2}$ compatible with the boundary conditions (generalized metrics)?)
- "T-duality" in higher dimensions and for (higher) gauge theories ($n > 2$)