Noncommutative Gauge Theories on D-Branes in Non-Geometric Backgrounds

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

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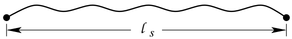
Outline

- Introduction/Motivation
- Review: D-branes in constant B-fields
- Non-geometric backgrounds:
  Expectations from topological T-duality
- Twisted tori & D-branes in T-folds
- Doubled twisted tori & D-branes in R-folds

with Chris Hull [arXiv:1902.xxxxx]

String Geometry



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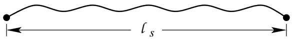

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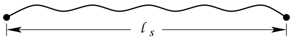

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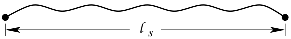

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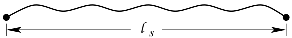

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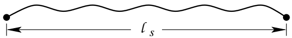

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- Not all spacetime geometries are ordinary geometric spaces, e.g. noncommutative spaces can arise as decoupling limits
- Use effective field theories as probes of geometry: Introduce D-branes and take decoupling limit $\Longrightarrow$ Noncommutative worldvolume gauge theories in an NS–NS B-field background

Open String Dynamics in Constant B-Fields

(Douglas & Hull '97; Ardalan, Arfaei & Sheikh-Jabbari '98; Chu & Ho '98; Schomerus '99; Seiberg & Witten '99; . . .)

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$$G = -(2\pi\,\alpha')^2\,B\,g^{-1}\,B \qquad , \qquad \theta = B^{-1}$$

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- Extend to curved backgrounds and non-constant B with H-flux $H = \mathrm{d}B \neq 0$ (Cornalba & Schiappa '01; Herbst, Kling & Kreuzer '01)

Noncommutative Yang-Mills Theory

- Open string interactions in scattering amplitudes captured by Moyal-Weyl star-product:

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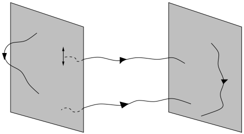

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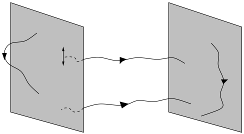

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- Effective Yang-Mills coupling in Dp-brane gauge theory:

$$g_{\rm YM}^2 = \frac{(2\pi)^{p-2}}{(\alpha')^{(3-p)/2}}\ g_s\ \mathrm{e}^{\,\phi}\left(\frac{\det(g + 2\pi\,\alpha'\,B)}{\det g}\right)^{1/2}$$

Finite in decoupling limit if $g_s\ \mathrm{e}^{\,\phi} \sim \epsilon^{(3-p+r)/4}$, $\ r = \mathrm{rank}(B)$

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- Noncommutative gauge theory inherits this T-duality symmetry
- Refinement of topological T-duality via Morita equivalence of noncommutative tori: $K(\,T^p_\theta) = K(\,T^p_{\theta'})$

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- T^3 with H-flux gives geometric and non-geometric fluxes via T-duality (Hull '05; Shelton, Taylor & Wecht '05; Dabholkar & Hull '06; ...)

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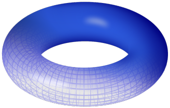
- **Goal:** Understand worldvolume gauge theories in these non-geometric backgrounds [extending (Lowe, Natase & Ramgoolam '03; Ellwood & Hashimoto '06; Grange & Schäfer-Nameki '07)]; compare with noncommutative/nonassociative closed string geometry (Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen, Deser, Lüst, Plauschinn & Rennecke '11; Mylonas, Schupp & Sz '12; Freidel, Leigh & Minic '17; . . .)

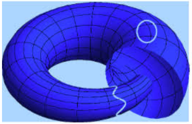
Geometry vs. Non-Geometry

Twisted torus



$$\downarrow \qquad \xrightarrow{T_2} \qquad \downarrow$$

T-fold



S^1 $\qquad$ S^1

Patching: Diffeos

Patching: T-duality

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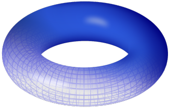

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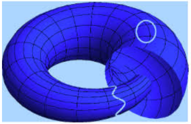


S^1 S^1

$\downarrow \mathrm{T}_x$

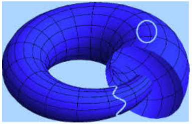
R-fold

Geometry vs. Non-Geometry

(T^3, H-flux): $[H] = m$

$\uparrow \mathrm{T}_1$

Nilfold (f) $\xrightarrow{\mathrm{T}_2}$ T-fold (Q)



Nilfold $\xrightarrow[m]{S^1}$ T^2

T-fold $\longrightarrow$ S^1

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R-fold (R)

Expectations from Topological T-Duality

(Mathai & Rosenberg '04; Bouwknegt, Hannabuss & Mathai '06;

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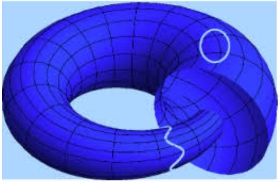

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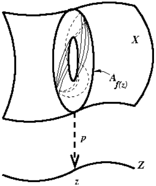

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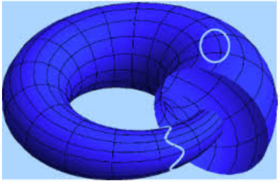

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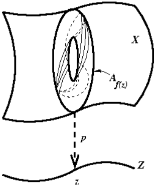

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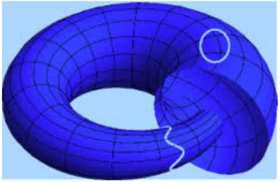

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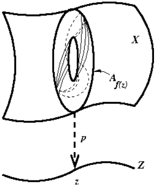

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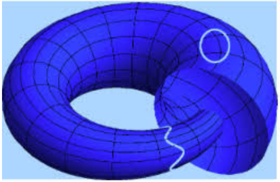

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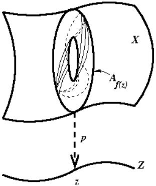

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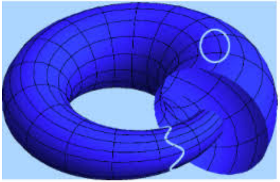

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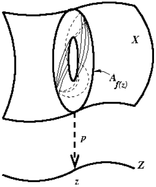

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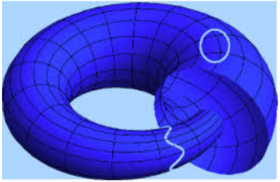

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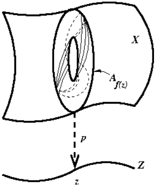

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- R-flux $(d=3)$: $\widehat{\mathcal{A}} = \mathcal{K}\big(L^2(\widehat{T}^3)\big) \rtimes_{u_\phi} \widehat{T}^3 =$ nonassociative 3-torus T^3_ϕ , $\phi \in Z^3\big(\widehat{T}^3, U(1)\big)$ associated to H R-fold

The Twisted Torus as a T^2-Bundle

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- Dependence of moduli determined by $\gamma(x) \ = \ \exp(x\, M) \in O(2,2)$ with monodromy $\mathcal{M}(\gamma) \ = \ \gamma(0)\, \gamma^{-1}(1) \ = \ \exp(M)$ in:

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$$\mathrm{d}s_X^2 = (2\pi\,r\,\mathrm{d}x)^2 + \frac{A}{\tau_2}\,\big|\mathrm{d}y^1 + \tau\,\mathrm{d}y^2\big|^2 \quad , \quad \tau(x) = \gamma(x)[\tau^\circ]$$

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- For $\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$: $\tau(x+1) = \mathcal{M}\big[\tau(x)\big] = \dfrac{a\, \tau(x) + b}{c\, \tau(x) + d}$

The Twisted Torus as a Group Quotient

- $X = G_{\mathbb{R}}/G_{\mathbb{Z}}$, with generators J_1, J_2, J_x:

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- Conjugacy classes of $SL(2, \mathbb{Z})$:
  1. Parabolic: $\mathrm{Tr}(\mathcal{M}) = 2$
  2. Elliptic: $\mathrm{Tr}(\mathcal{M}) < 2$
  3. Hyperbolic: $\mathrm{Tr}(\mathcal{M}) > 2$

The Twisted Torus and T-Duality

- Gauge theory: Wrap D1-brane on (torsion) y^1-cycle, at $y^2 = 0$ and any $x \in S^1$, and follow its orbits under T-duality using Buscher rules

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\begin{aligned}
g &= (2\pi\, r)^2\, \mathrm{d}x^2 + \frac{\tau_2(x)}{|\tau(x)|^2} \left(A \left(\mathrm{d}y^1\right)^2 + \frac{(2\pi\,\alpha')^2}{A} \left(\mathrm{d}y^2\right)^2 \right) \\
B &= \frac{\tau_1(x)}{|\tau(x)|^2}\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \\
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- Apply open-closed string transformation:

$$G = \frac{A}{\tau_2(x)}\,(\mathrm{d}y^1)^2 + \frac{(2\pi\,\alpha')^2}{A\,\tau_2(x)}\,(\mathrm{d}y^2)^2$$

$$\theta = \tau_1(x)\,\partial_{y^1} \wedge \partial_{y^2}$$

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- There is a consistent decoupling limit of the D2-brane on the T-fold with $\alpha', A, \tau_2^\circ, g_s \sim \epsilon^{1/2}$ such that as $\epsilon \longrightarrow 0$:

$$G = \big(2\pi\, r_1\, \mathrm{d}y^1\big)^2 + \big(2\pi\, r_2\, \mathrm{d}y^2\big)^2$$

$$\theta(x) = m\,x \qquad , \qquad g_{\mathrm{YM}}^2 = 2\pi\, \bar{g}_s$$

Parabolic Twists and Noncommutative Gauge Theory

- Since $\partial_{y^a}\theta = 0$, Kontsevich formula gives star-product:

$$f \star g = \cdot \exp\Big(\tfrac{\mathrm{i}}{2}\, m\, x\, (\partial_{y^1} \otimes \partial_{y^2} - \partial_{y^2} \otimes \partial_{y^1})\Big)(f \otimes g)$$

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- Open strings see conventional geometric T^3 with non-geometric noncommutativity $\theta(x)$!
(cf. Morita equivalence symmetry of noncommutative Yang-Mills theory is inherited from T-duality in decoupling limit)

Elliptic Twists

- Monodromies of finite order: $\mathcal{M} = U \begin{pmatrix} \cos(m\,\vartheta) & \sin(m\,\vartheta) \\ -\sin(m\,\vartheta) & \cos(m\,\vartheta) \end{pmatrix} U^{-1}$

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- Decoupled open string noncommutative geometry:

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$$\theta(x) = \tan\left(\tfrac{m\,\pi}{2}\,x\right) \qquad , \qquad g_{\mathrm{YM}}(x)^2 = 2\pi\,\bar{g}_s\,\Big|\cos\left(\tfrac{m\,\pi}{2}\,x\right)\Big|$$

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- Open strings now simultaneously probe both a non-geometric and a noncommutative space !

The Doubled Twisted Torus

- Double the twisting: $\gamma(x,\widetilde{x}) \; = \; \exp(x\,M)\,\exp(\widetilde{x}\,\widetilde{M}) \in O(2,2)$

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- Doubled metric $\mathrm{d}s^2_{\mathscr{X}} = \mathcal{H}_{IJ}\,\mathrm{d}\mathbb{X}^I\,\mathrm{d}\mathbb{X}^J$, $\mathcal{H} \in O(3,3)/O(3)\times O(3)$:

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- T-duality realized linearly by $O(3,3;\mathbb{Z})$-transformations, use to follow orbits of D-branes in doubled twisted torus geometry
  (Lawrence, Schulz & Wecht '06; Albertsson, Kimura & Reid-Edwards '08)

R-Flux and Noncommutative Gauge Theory

Background	Dp-brane	x	y^1	y^2	$\widetilde{x}$	$\widetilde{y}_1$	$\widetilde{y}_2$
H-flux	D0-brane	$-$	$-$	$-$	$\times$	$\times$	$\times$
f-flux	D1-brane	$-$	$\times$	$-$	$\times$	$-$	$\times$
Q-flux	D2-brane	$-$	$\times$	$\times$	$\times$	$-$	$-$
R-flux	D3-brane	$\times$	$\times$	$\times$	$-$	$-$	$-$

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- Decoupling limit in R-fold additionally requires $r \sim \epsilon^{1/2}$, with open string noncommutative geometry:

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\begin{aligned}
G_R &= (2\pi\, \bar{r}_x\, \mathrm{d}x)^2 + G_{\mathrm{D2}}\big|_{x \to \widetilde{x}} \\
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- Noncommutative D3-brane gauge theory in $\mathscr{X}$ returns to itself under $\widetilde{x}\longmapsto\widetilde{x}+1$ up to Morita equivalence