On generic superconformal field theories on K3

Quantum structure of spacetime

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Plan:

1. Superconformal field theories on K3
2. Refining the conformal field theoretic elliptic genus
3. Elliptic genera in geometry
4. Results

[W17] *Hodge-elliptic genera and how they govern K3 theories*, accepted for publication by Comm. Math. Phys.; arXiv:1705.09904 [hep-th]

[Taormina/W15] *A twist in the M_{24} moonshine story*, Confluentes Mathematici 7, 1 (2015), 83-113; arXiv:1303.3221 [hep-th]

[Taormina/W13] *Symmetry-surfing the moduli space of Kummer K3s*, Proceedings of Symposia in Pure Mathematics 90 (2015), 129-153; arXiv:1303.2931 [hep-th]

[Taormina/W11] *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP **1308**:152 (2013); arXiv:1107.3834 [hep-th]

1. Superconformal field theories on K3 - what they are

M: a compact Calabi-Yau D-fold

parameters of non-linear sigma-models on M:

complex structure, Kähler-Einstein metric, B-field

Result:

- **[Casher/Englert/Nicolai/Taormina85, Narain86]**
  The moduli space $\mathcal{M}_{T^D}$ of non-linear sigma models on a complex D-torus is
  $$\mathcal{M}_{T^D} = {}_{\mathrm{O}(D,D;\mathbb{Z})}\backslash^{\mathrm{O}(D,D;\mathbb{R})}\!/_{\mathrm{O}(D)\times\mathrm{O}(D)}.$$
- **[Seiberg88, Cecotti90; Aspinwall/Morrison94; Nahm/W00]**
  The moduli space $\overline{\mathcal{M}_{\mathrm{K3}}}$ of non-linear sigma models on K3 is
  $$\overline{\mathcal{M}_{\mathrm{K3}}} = {}_{\mathrm{O}^+(4,20;\mathbb{Z})}\backslash^{\mathrm{O}(4,20;\mathbb{R})}\!/_{\mathrm{O}(4)\times\mathrm{O}(20)}.$$

1. Superconformal field theories - what we assume

Notations and assumptions:

$\mathbb{H}$: Neveu-Schwarz sector of a unitary 2-dimensional $N=(2,2)$ superconformal field theory, such that

- the spectral flow isomorphically maps $\mathbb{H}$ to the Ramond sector (space-time supersymmetry)
- the central charges are $c=\overline{c}=3D$ with $D\in\mathbb{N}$
- all eigenvalues of J_0, $\overline{J}_0$ and $H-\overline{H}$ are integral, where $2H:=2L_0-J_0,\ 2\overline{H}:=2\overline{L}_0-\overline{J}_0$

With $\tau, z\in\mathbb{C}$, $\mathrm{Im}(\tau)>0$, $q:=\exp(2\pi i\tau)$, $y:=\exp(2\pi iz)$,

WITTEN GENUS:

$$\chi(\mathbb{H}):=\mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\overline{J}_0}q^H\overline{q}^{\overline{H}}\right)$$

CONFORMAL FIELD THEORETIC ELLIPTIC GENUS:

$$\mathcal{E}_{\mathrm{CFT}}(\mathbb{H};\tau,z):=\mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\overline{J}_0}y^{J_0-c/6}q^H\overline{q}^{\overline{H}}\right)$$

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1. Defining K3 theories

Definition

A K3 THEORY is a superconformal field theory as above at $c = 6$ with Witten index $\chi(\mathbb{H}) = 24$.

Results:

- **[Eguchi/Ooguri/Taormina/Yang89]**
  All K3 theories possess $N = (4,4)$ supersymmetry and have conformal field theoretic elliptic genus

$$\mathcal{E}_{\text{CFT}}(\mathbb{H};\tau,z) = 8\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2.$$

- **[Cecotti90; Aspinwall/Morrison94; Nahm/W00]**
  There is an 80-dimensional moduli space of K3 theories, one connected component being $\mathcal{M}_{\text{K3}}$ with

$$\overline{\mathcal{M}_{\text{K3}}} = \mathrm{O}^+(4,20;\mathbb{Z})\backslash \mathrm{O}(4,20;\mathbb{R}) / \mathrm{O}(4)\times\mathrm{O}(20).$$

2. The conformal field theoretic Hodge elliptic genera

Elliptic genus

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2. The conformal field theoretic Hodge elliptic genera

Hodge elliptic genera

With $\nu \in \mathbb{C}$, $u := \exp(2\pi i \nu)$,

- **[Kachru/Tripathy16]**

$$\mathcal{E}^{\text{HEG}}_{\text{CFT}}(\mathbb{H}; \tau, z, \nu) := \text{tr}_{\ker(\overline{H})}\left((-1)^{J_0 - \overline{J}_0} y^{J_0 - c/6} u^{\overline{J}_0 - c/6} q^{H} \right).$$

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- **[W17]**

  Let $\mathbb{H}_0$ denote the maximal $\mathbb{C}$-vector space such that, as a representation of $\{H, J_0, \overline{J}_0\}$, for every theory in the moduli space,

$$\mathbb{H}_0 \hookrightarrow \ker(\overline{H})$$

– the GENERIC SPACE OF STATES;

$$\mathcal{E}^{\mathrm{HEG},0}_{\mathrm{CFT}}(\mathbb{H}; \tau, z, \nu) := \mathrm{tr}_{\mathbb{H}_0}\left((-1)^{J_0 - \overline{J}_0} y^{J_0 - c/6} u^{\overline{J}_0 - c/6} q^{H} \right).$$

2. Prediction for the generic space of states of K3 theories

3 types of irreducible $N=4$ representations $\mathcal{H}_\bullet$ **[Eguchi/Taormina/87]**;
with $\chi_\bullet(\tau,z) := \mathrm{tr}_{\mathcal{H}_\bullet}\left((-1)^{J_0} y^{J_0-c/6} q^H\right)$, $\mathcal{H}^0_\bullet := \mathcal{H}_\bullet \cap \ker(H)$:

- vacuum $\mathcal{H}_0$ with $\chi_0(\tau,0)=-2$; $\mathcal{H}^0_0 \cong \mathbb{C}^2$.
- massless matter $\mathcal{H}_{mm}$ with $\chi_{mm}(\tau,0)=1$; $\mathcal{H}^0_{mm} \cong \mathbb{C}$.
- massive matter $\mathcal{H}_h$ $(h \in \mathbb{R}_{>0})$, $\chi_h(\tau,z) = q^h \widetilde{\chi}(\tau,z)$, $\chi_h(\tau,0)=0$; $\mathcal{H}^0_h = \{0\}$.

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Ansatz:

$$\mathbb{H} = \mathcal{H}_0 \otimes \overline{\mathcal{H}}_0 \oplus 20\, \mathcal{H}_{mm} \otimes \overline{\mathcal{H}}_{mm} \oplus \left(\bigoplus_{n\in\mathbb{N}_{>0}} \left[f_n \mathcal{H}_n \otimes \overline{\mathcal{H}}_0 \oplus \overline{f}_n \mathcal{H}_0 \otimes \overline{\mathcal{H}}_n\right]\right)$$
$$\oplus \left(\bigoplus_{\ell\in\mathbb{N}_{>0}} \left[g_\ell \mathcal{H}_\ell \otimes \overline{\mathcal{H}}_{mm} \oplus \overline{g}_\ell \mathcal{H}_{mm} \otimes \overline{\mathcal{H}}_\ell\right]\right) \oplus \bigoplus_{h,\overline{h}\in\mathbb{R}_{>0}} k_{h,\overline{h}} \mathcal{H}_h \otimes \overline{\mathcal{H}}_{\overline{h}}$$

where all f_n, $\overline{f}_n$, g_ℓ, $\overline{g}_\ell$, $k_{h,\overline{h}}$ are non-negative integers.

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- $\mathbb{H}_{\min} := \mathcal{H}_0\otimes\overline{\mathcal{H}}_0^0 \oplus 20\,\mathcal{H}_{mm}\otimes\overline{\mathcal{H}}_{mm}^0 \oplus \bigoplus_{\ell\in\mathbb{N}_{>0}} a_\ell\mathcal{H}_\ell\otimes\overline{\mathcal{H}}_{mm}^0 \quad \subset \mathbb{H}_0$
- [generic chiral algebra of K3 theories] = [$N=4$ superconformal algebra]
  $\Longleftrightarrow \forall\ell\in\mathbb{N}$, generically $f_\ell=0 \quad\Longleftrightarrow\quad \mathbb{H}_{\min}=\mathbb{H}_0$.

3. Topologically half-twisted σ-models - a math version?

M: a compact Calabi-Yau D-fold, $T := T^{1,0}M$;

for any bundle $E \to M$, $\Lambda_x E := \bigoplus_{k=0}^{\infty} x^k \Lambda^k E$, $S_x E := \bigoplus_{k=0}^{\infty} x^k S^k E$

Definition [Hirzebruch88, Witten88]

$$\mathbb{E}_{q,-y} := y^{-\frac{D}{2}} \Lambda_{-y} T^* \otimes \bigotimes_{n=1}^{\infty} \left[\Lambda_{-yq^n} T^* \otimes \Lambda_{-y^{-1}q^n} T \otimes S_{q^n} T^* \otimes S_{q^n} T \right]$$

$$= y^{-\frac{D}{2}} \bigoplus_{\ell,m} q^\ell (-y)^m \mathcal{T}_{\ell,m},$$

$$\mathcal{E}(M;\tau,z) := y^{-\frac{D}{2}} \sum_{\ell,m} q^\ell (-y)^m \sum_j (-1)^j \dim H^j(M, \mathcal{T}_{\ell,m})$$

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$$\mathcal{E}(K3;\tau,z) = 8\left(\frac{\vartheta_2(\tau,z)}{\vartheta_2(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_3(\tau,z)}{\vartheta_3(\tau,0)}\right)^2 + 8\left(\frac{\vartheta_4(\tau,z)}{\vartheta_4(\tau,0)}\right)^2.$$

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In general, using D copies of $bc - \beta\gamma$ systems

with modes $\beta_n^{(j)}$, $\gamma_n^{(j)}$, $b_n^{(j)}$, $c_n^{(j)}$, $n \in \mathbb{Z}$, $j \in \{1,\dots,D\}$:

$S_{q^n}(\mathbb{C}^D) \cong \mathrm{Sym}(\gamma_n^{(1)},\dots,\gamma_n^{(D)})$, $S_{q^n}((\mathbb{C}^D)^*) \cong \mathrm{Sym}(\beta_n^{(1)},\dots,\beta_n^{(D)})$,

$\Lambda_{-yq^n}(\mathbb{C}^D) \cong \Lambda(c_n^{(1)},\dots,c_n^{(D)})$, $\Lambda_{-y^{-1}q^n}((\mathbb{C}^D)^*) \cong \Lambda(b_n^{(1)},\dots,b_n^{(D)})$.

3. The chiral de Rham complex in a nutshell

Definition **[Malikov/Schechtman/Vaintrob99]**
CHIRAL DE RHAM COMPLEX Ω_M^{ch}: sheaf of super-VOAs over M,
for any holomorphic coordinate chart $U \subset M$:

$$\Omega_M^{\text{ch}}(U) := D \text{ copies of a } bc-\beta\gamma\text{-system.}$$

Theorem **[Malikov/Schechtman/Vaintrob99; Borisov/Libgober00]**
$H^*(M, \Omega_M^{\text{ch}})$ (sheaf cohomology) is a topological $N = 2$ superconformal VOA.
Ω_M^{ch} is filtered with associated graded $\mathbb{E}_{q,-y}$ ($q \leftrightarrow H$, $y \leftrightarrow J_0$).

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Consequence:

$$\mathcal{E}(M;\tau,z) = y^{-\frac{D}{2}} \sum_j (-1)^j \underbrace{\operatorname{tr}_{H^j(M,\Omega_M^{ch})}\left((-y)^{J_0} q^H\right)}_{\neq \text{ gr-dim}\left(H^j(M,\mathbb{E}_{q,-y})\right), \text{ in general}}$$

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Result [Kapustin05]
With $\ker(\overline{H}) \subset \mathbb{H}$, the topologically half-twisted sigma model on M: the large volume limit of $\ker(\overline{H})$ is $H^*(M, \Omega^{ch}_M)$.

3. The chiral de Rham complex in a nutshell

Definition [Malikov/Schechtman/Vaintrob99]
CHIRAL DE RHAM COMPLEX Ω_M^{ch}: sheaf of super-VOAs over M,
for any holomorphic coordinate chart $U \subset M$:
$\Omega_M^{\text{ch}}(U) := D$ copies of a $bc - \beta\gamma$-system.

Theorem [Malikov/Schechtman/Vaintrob99; Borisov/Libgober00]
$H^*(M, \Omega_M^{\text{ch}})$ (sheaf cohomology) is a topological $N=2$ superconformal VOA.
Ω_M^{ch} is filtered with associated graded $\mathbb{E}_{q,-y}$ ($q \leftrightarrow H$, $y \leftrightarrow J_0$).

Consequence:

$$\mathcal{E}(M;\tau,z) = y^{-\frac{D}{2}} \sum_j (-1)^j \underbrace{\operatorname{tr}_{H^j(M,\Omega_M^{\text{ch}})}\left((-y)^{J_0} q^H\right)}_{\neq \text{gr-dim}\left(H^j(M,\mathbb{E}_{q,-y})\right),\text{ in general}}$$

Result [Kapustin05]
With $\ker(\overline{H}) \subset \mathbb{H}$, the topologically half-twisted sigma model on M: the large volume limit of $\ker(\overline{H})$ is $H^*(M, \Omega_M^{\text{ch}})$. Hence $\mathcal{E}(M;\tau,z) = \mathcal{E}_{\text{CFT}}(\mathbb{H};\tau,z)$.

3. Hodge elliptic genera

M, $\mathbb{H}$, $\mathbb{E}_{q,-y} = y^{-\frac{d}{2}} \bigoplus_{\ell,m} q^\ell (-y)^m \mathcal{T}_{\ell,m}$ as before, $\nu \in \mathbb{C}$, $u := \exp(2\pi i \nu)$.

Hodge elliptic genera

- **[Kachru/Tripathy16]**

$$\mathcal{E}^{\mathrm{HEG}}_{\mathrm{CFT}}(\mathbb{H};\tau,z,\nu) := \mathrm{tr}_{\ker(\overline{H})}\left((-1)^{J_0-\overline{J}_0} y^{J_0-c/6} u^{\overline{J}_0-c/6} q^H\right).$$

- **[W17]**

$$\mathcal{E}^{\mathrm{HEG},0}_{\mathrm{CFT}}(\mathbb{H};\tau,z,\nu) := \mathrm{tr}_{\mathbb{H}_0}\left((-1)^{J_0-\overline{J}_0} y^{J_0-c/6} u^{\overline{J}_0-c/6} q^H\right).$$

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4. Results

Results – if M is a K3 surface and $\mathbb{H}$ belongs to a K3 theory:

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Open:

- Is any VOA structure of $\mathbb{H}_0$ compatible with the M_{24}-action?
- Is M_{24} generated by symmetry surfing, as suggested in **[Taormina/W11+ $\cdots$]**?

Thank you
for your attention!