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On noncommutative spacetimes and gauge groups

Paolo Aschieri
Università del Piemonte Orientale, Alessandria, Italy

Based on joint work with Landi, Pagani
Bieliavsky, Pagani, Schenkel

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NC spacetime $\longleftrightarrow$ quantum gravity

spacetime $\longleftrightarrow$ arena for quantum dynamics

• NC arena may provide simpler descriptions than a commutative one. E.g.

$$\mathcal{L} = \frac{F^2 - \theta \tilde{F} F \tilde{F}}{1 - \theta F} \qquad \widehat{\mathcal{L}} = -\widehat{F}^2$$

for slowly varying fields. The two actions are the same under Seiberg-Witten map

$$\widehat{A} = \widehat{A}(A, \theta)$$

Gauge equivalence classes of $\widehat{A}$ are in 1-1 correspondence with those of A.

• T-duality symmetries, map commutative to NC gauge theories. T-duality acts within NC gauge theories.

In these examples motivated from strings the gauge groups are $U(1)$ and $U(n)$. What about $SU(n)$, $SO(n)$,...

Quantum groups are defined for any simple Lie group. Are there corresponding quantum group gauge theories?

- One way out [Wess Group] is SW map.
The NC $\widehat{A} = \widehat{A}(A, \theta)$ is a power series in the generators T^a of the classical gauge group G; $\widehat{A}$ is UEA valued.

Is it possible to give def. indep. from classical fields? More intrinsic def.?

This question gains further relevance thinking about the key role in diff. geometry of the notion of **principal bundle** and its associated gauge group and connection. One would expect quantum principal bundles with quantum structure groups and quantum gauge groups.

rmk. NC Principal bundles are less understood than NC vector bundles

Princ. G-Bundle

If the bundle $P \longrightarrow M$ is a principal G-bundle:
The G-action on P, $P \times G \to P$ is fiber preserving
The G-action is free on P and
The G-action is transitive on the fibers
$M \simeq P/G$

i.e., the map

$$\begin{aligned} P \times G &\longrightarrow P \times_M P \\ (p, g) &\longmapsto (p, pg) \end{aligned}$$ is injective and surjective

Description in terms of algebras

$A \sim C^\infty(P) \qquad A \otimes A \;\sim\; C^\infty(P \times P)$ (Completion $\hat{\otimes}$ is understood or consider A the coordinate ring of an affine variety).

$H \sim C^\infty(G) \qquad A \otimes H \;\sim\; C^\infty(P \times G)$

$P \times G \to P$ dualizes to $A \longrightarrow A \otimes H$

$(p, g) \mapsto pg \qquad a \mapsto \delta^A(a) = a_0 \otimes a_1 \qquad (a_0 \otimes a_1)(p, g) = a(pg)$

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$B \ \sim \ C^\infty(M) \ \simeq \ C^\infty(P/G)$ i.e. B is the subalgebra of $A \ \sim \ C^\infty(P)$ of functions constant along the fibers

$B = A^{coH} = \{b \in A, \, \delta^A(b) = b \otimes 1\} \subset A$

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Then $P \times G \longrightarrow P \times_M P$ is bijective iff

$$\chi : A \otimes_B A \longrightarrow A \otimes H$$

$$a \otimes_B a' \longmapsto a a'_0 \otimes a'_1 \text{ is bijective}$$

A is an H-comodule algebra because of the compatibility: $\delta^A(a\tilde{a}) = \delta^A(a)\delta^A(\tilde{a})$.

Def. of Hopf-Galois extension

Let H be a Hopf algebra and A be an H-comodule algebra,

$B = A^{coH} \subset A$ is a Hopf-Galois extension if $\chi : A \otimes_B A \to A \otimes H$ is a bijection

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Equivariance property of χ

If H and A are commutative alg. then χ is an algebra map, this is no more true in the NC case

We show that χ is compatible with the H-coaction (the G-action).

A is an H-comodule, we write $A \in \mathcal{M}^H$
H is also an H-comodule with the Ad-action of H on H

$$Ad : H \to H \otimes H \qquad\qquad G \times G \to G$$

$$h \mapsto h_2 \otimes S(h_1)h_3 \qquad\qquad (g, g') \mapsto {g'}^{-1} g\, g'$$

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Equivariance property of χ

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The canonical map χ is compatible with the H-coaction (the G-action).

A is an H-comodule, we write $A \in \mathcal{M}^H$
H is also an H-comodule with the Ad-action of H on H

$$\begin{aligned} Ad : H &\to H \otimes H \\ h &\mapsto h_2 \otimes S(h_1)h_3 \end{aligned} \qquad \begin{aligned} G \times G &\to G \\ (g, g') &\mapsto g'^{-1} g\, g' \end{aligned}$$

Since $A, H \in \mathcal{M}^H$ then also $A \otimes H \in \mathcal{M}^H$, $A \otimes A \in \mathcal{M}^H$, $A \otimes_B A \in \mathcal{M}^H$.

It is now easy to show that χ is an H-comodule map (it is equivariant).

Moreover χ is compatible with multiplication of $A \otimes H$ and of $A \otimes_B A$ from the left with elements of A, i.e. it is a left A-module map