NCG for Spacetime

by John Barrett 16.1.2019

- Finite spectral triples for the fuzzy torus,
  JB, James Gaunt 99%
- Spectral estimators for finite NCGs
  JB, Paul Druce, Lisa Glaser arxiv:1902.03590

Ideas

- Space-Time is a NCG
- Finite # modes per unit volume

  — Planck scale
- Dirac operator specifies geometry

  — as in particle physics

Fuzzy spaces

Manifold M $\rightsquigarrow$ Algebra $\mathcal{A} = C^\infty(M)$

Hilbert space $\mathcal{H} = L^2(M)$

Fuzzy space $\rightsquigarrow$ $\mathcal{A} \subseteq M_N(\mathbb{C})$

$\mathcal{H}$: bimodule over $\mathcal{A}$

Unitary structures

$\mathcal{H}$ is a Hilbert space $(\cdot,\cdot)$

$\mathcal{A}$ is a $*$-algebra

$$(a^* \psi b^*, \phi) = (\psi, a\phi b)$$

$J : \mathcal{H} \to \mathcal{H}$ antiunitary "real structure"

$$J^2 = \epsilon = \pm 1$$

$$J(a\psi b) = b^* (J\psi) a^*$$

Geometry via Laplace

Fuzzy S^2 : $L_1, L_2, L_3 \in A$ $[L_1, L_2] = L_3$ etc.,

$$\Delta\psi = -\sum_{i=1}^{3} [L_i, [L_i, \psi]]$$

spin j	0	1	2	......	$N-1$
$\lambda = j(j+1)$	0	2	6	......	$N^2 - N$

— same as S^2 with cutoff.

Fuzzy torus

- **Algebra** $\quad U, V \in \mathrm{End}(\mathcal{H}), \quad U^* U = V^* V = 1.$

$$\mathcal{A} = \langle U, V \rangle$$

- **Geometry**

$$XY = Q\, YX \quad , \quad X, Y \in \langle U, V \rangle$$

$$\Delta_{X,Y} \psi = \frac{-1}{(Q^{1/2} - Q^{-1/2})^2} \left([X, [X^*, \psi]] + [Y, [Y^*, \psi]] \right)$$

Clock and shift

In $M_N(\mathbb{C})$, $\quad q^N = 1$ primitive

$$C = \begin{pmatrix} 1 & \cdot & \cdot & & \cdot \\ \cdot & q & \cdot & & \\ \cdot & \cdot & q^2 & & \\ & & & \ddots & \\ \cdot & & & & q^{N-1} \end{pmatrix} \qquad S = \begin{pmatrix} \cdot & 1 & & & \\ & \cdot & 1 & & \\ & & \cdot & 1 & \\ & & & \ddots & \ddots \\ 1 & & & & \cdot \end{pmatrix}$$

$$CS = qSC$$

Example: Square Torus

Algebra $\mathcal{H} = M_N(\mathbb{C})$, $J = *$

$U = C$, $V = S$, $\mathcal{A} = \langle U, V \rangle = M_N(\mathbb{C})$

Geometry $X = U$, $Y = V$

$$[k]_q = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}$$

$\Rightarrow$ Eigenvalues of Δ

$$\lambda_{k,l} = [k]_q^2 + [l]_q^2, \qquad k, l \in \mathbb{Z}_N$$

c.f. commutative $\lambda_{k,l} = k^2 + l^2$, $\quad k, l \in \mathbb{Z}$

Non-square geometry

$$X = q^{-ab/2} U^a V^b , \quad Y = q^{-cd/2} U^c V^d$$

$$XY = q^{ad-bc} YX$$

$$\lambda_{k,\ell} = \left([a\ell - bk]_q^2 + [dk - c\ell]_q^2 \right) / [ad-bc]_q^2$$

Commutative analogue:

$$\Delta = \frac{-1}{(ad-bc)^2} \left[(b^2+d^2) \frac{\partial}{\partial\theta^2} - 2(ab+cd)\frac{\partial^2}{\partial\theta\partial\phi} + (a^2+c^2)\frac{\partial^2}{\partial\phi^2} \right]$$

Real spectral triples

$$(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$$

$\mathcal{A}$: $*$-algebra $\qquad$ $\mathcal{H}$: Hilbert space

Operators on $\mathcal{H}$:

D Dirac operator

J Real structure

Γ Chirality

Examples : Manifold , fuzzy space

Dirac for fuzzy sphere

Grosse–Presnajder

$$\mathcal{H} = \mathbb{C}^2 \otimes M_n(\mathbb{C}) \qquad (\text{or } \mathbb{C}^4)$$

$$D\psi = \gamma^1 [L_1, \psi] + \gamma^2 [L_2, \psi] + \gamma^3 [L_3, \psi] + \psi$$

L_i : generators of $su(2)$.

Eigenvalues

Spin j	$1/2$	$3/2$	$5/2$	$\cdots$	$N - 1/2$
$\lambda^2 = j(j+1) + 1/4$	1	4	9	$\cdots$	N^2

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Dirac for fuzzy torus

$$\mathcal{H} = \mathbb{C}^4 \otimes M_N(\mathbb{C})$$

$$D\psi = \frac{1}{Q^{1/4} - Q^{-1/4}} \sum_{i=1}^{4} \gamma^i \, [K_i, \psi]$$

$$+ \frac{1}{Q^{1/4} + Q^{-1/4}} \sum_{i<j<k} \gamma^i \gamma^j \gamma^k \, \{K_{ijk}, \psi\}$$

$$K_1 = K_{234} = -\frac{1}{4}(X + X^*), \quad K_2 = -K_{134} = -\frac{i}{4}(X^* - X)$$

$$K_3 = -K_{124} = \frac{1}{4}(Y + Y^*), \quad K_4 = K_{123} = \frac{i}{4}(Y^* - Y)$$

Spectrum of square Torus Dirac

$$X = U = C, \quad Y = V = S$$

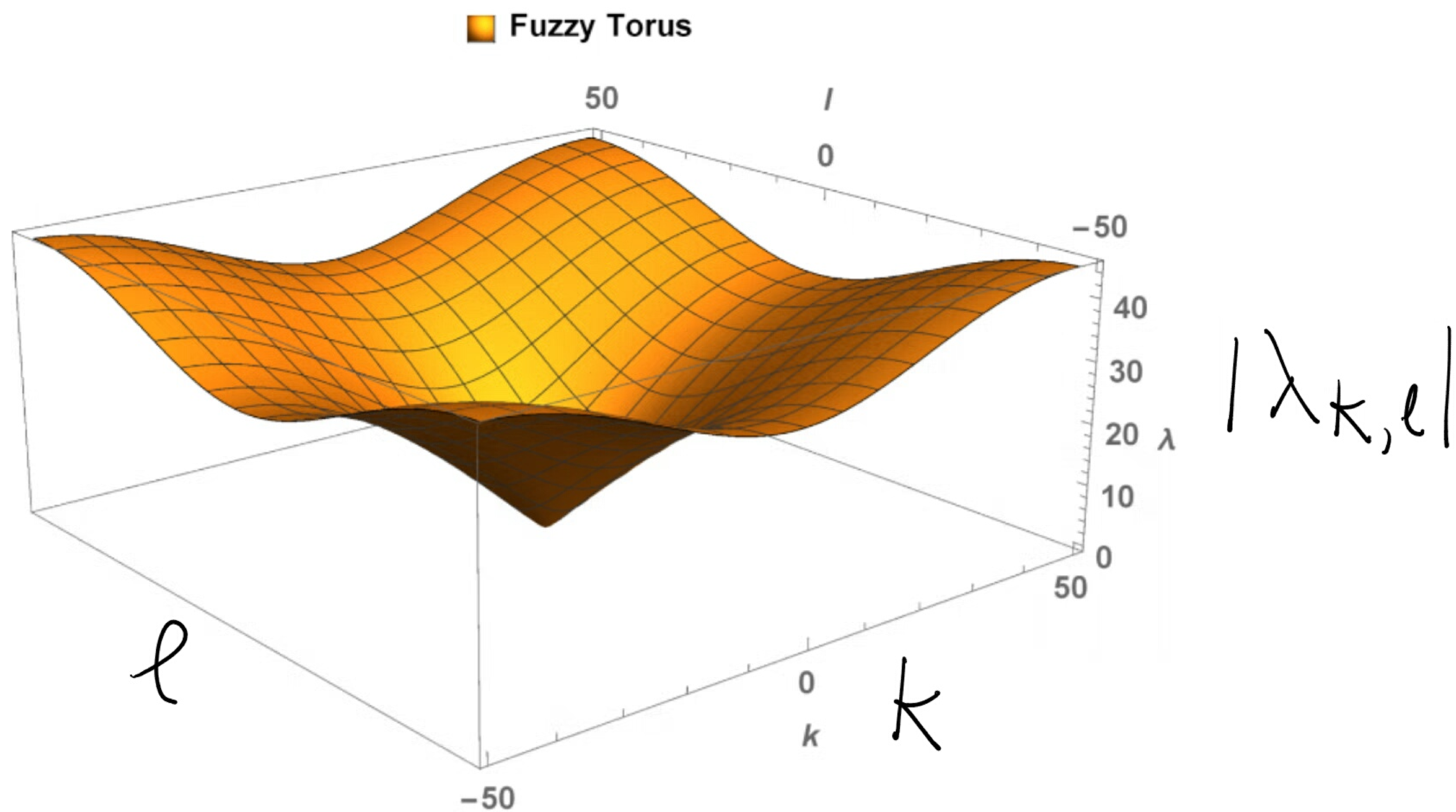
$$\lambda^2_{k,\ell} = [k]_q^2 + [\ell]_q^2$$

– but now $k, \ell \in \mathbb{Z} + \frac{1}{2}$

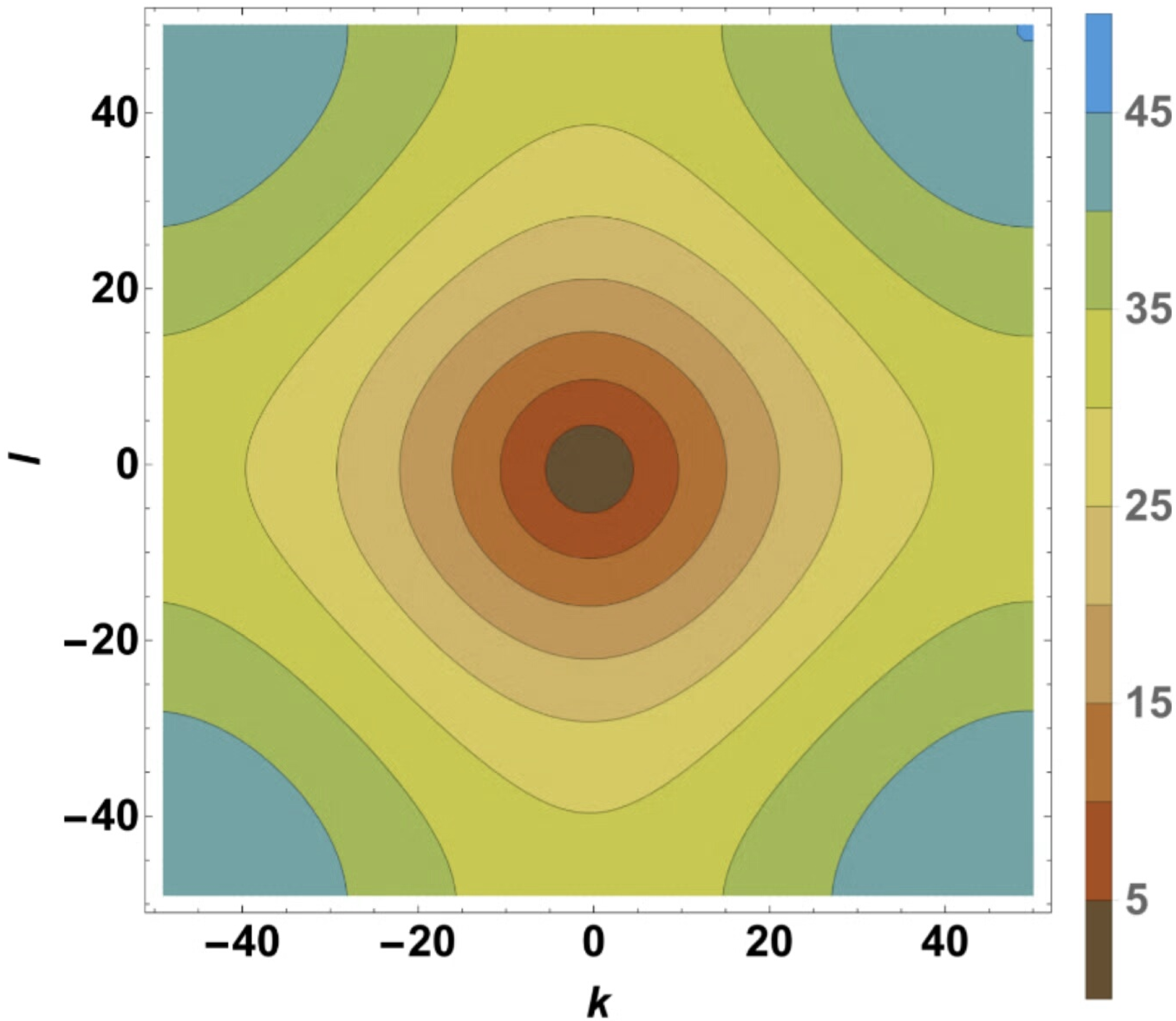
Spin structure $\quad \sigma_c = (1,1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$

"canonical spin structure"

Fuzzy torus eigenvalues $\quad N = 100$



Contour plot

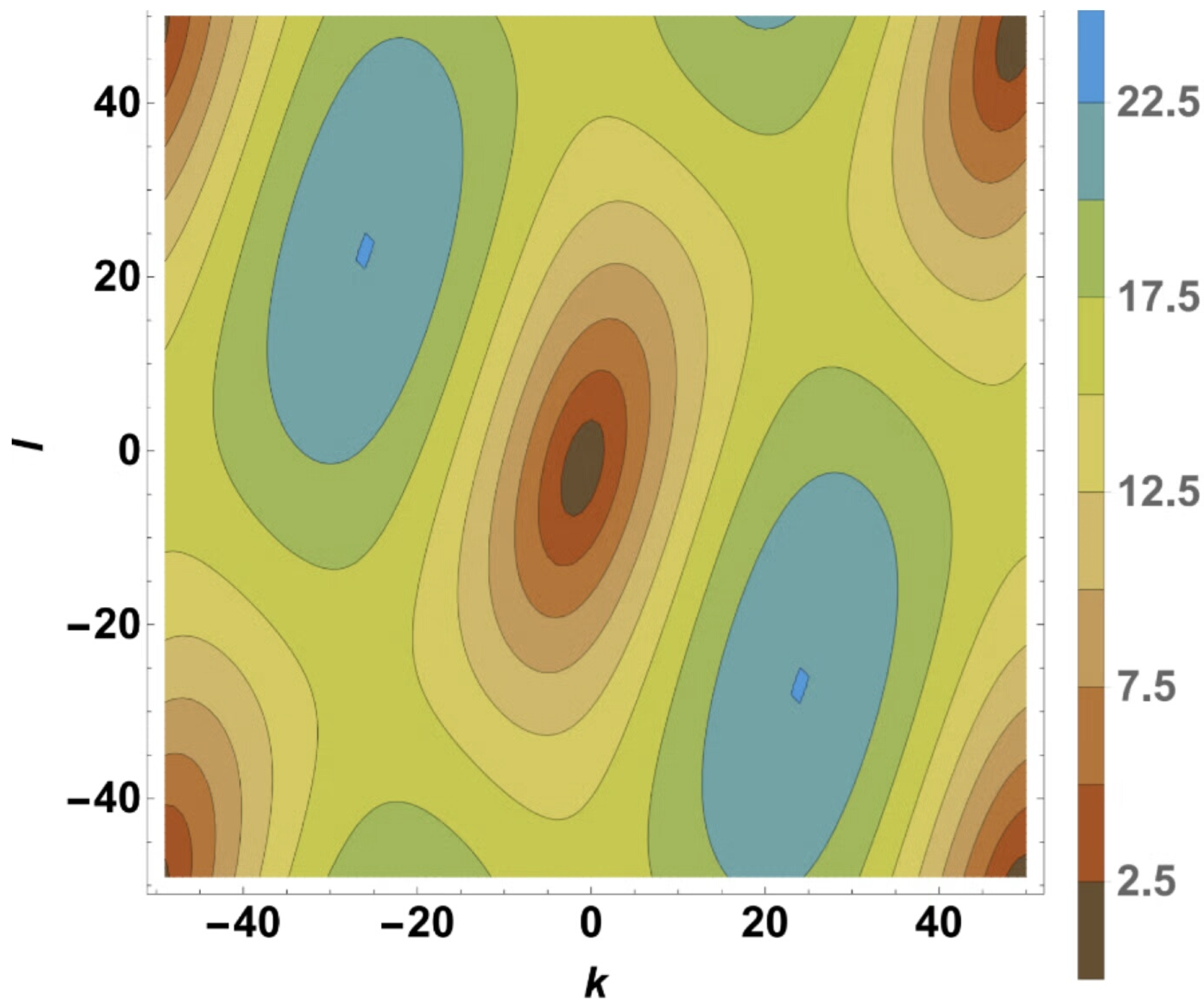


Non-square Torus

$$X = q^{-1/2} C S$$

$$Y = S^2$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$



All spin structures

$$\mathcal{H} = M_{4N}(\mathbb{C}) \qquad CS = q^{1/4} SC$$

$$U = C^2, \quad V = S^2 \qquad UV = qVU, \quad q^N = 1$$

$$\mathcal{A} = \langle U, V \rangle \subset M_{4N}(\mathbb{C})$$

covering map

Deck transformation group: $G = \langle U^N, V^N \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\mathcal{H} = \bigoplus_{\chi} \mathcal{H}_\chi, \qquad \chi : G \to \{\pm 1\} \subset \mathbb{C}$$

On X_χ: $\sigma = \sigma_c + \chi$

$\chi \in H^1(T^2; \mathbb{Z}_2)$

Spectral variance

$\mathcal{D}\psi = \lambda\psi$

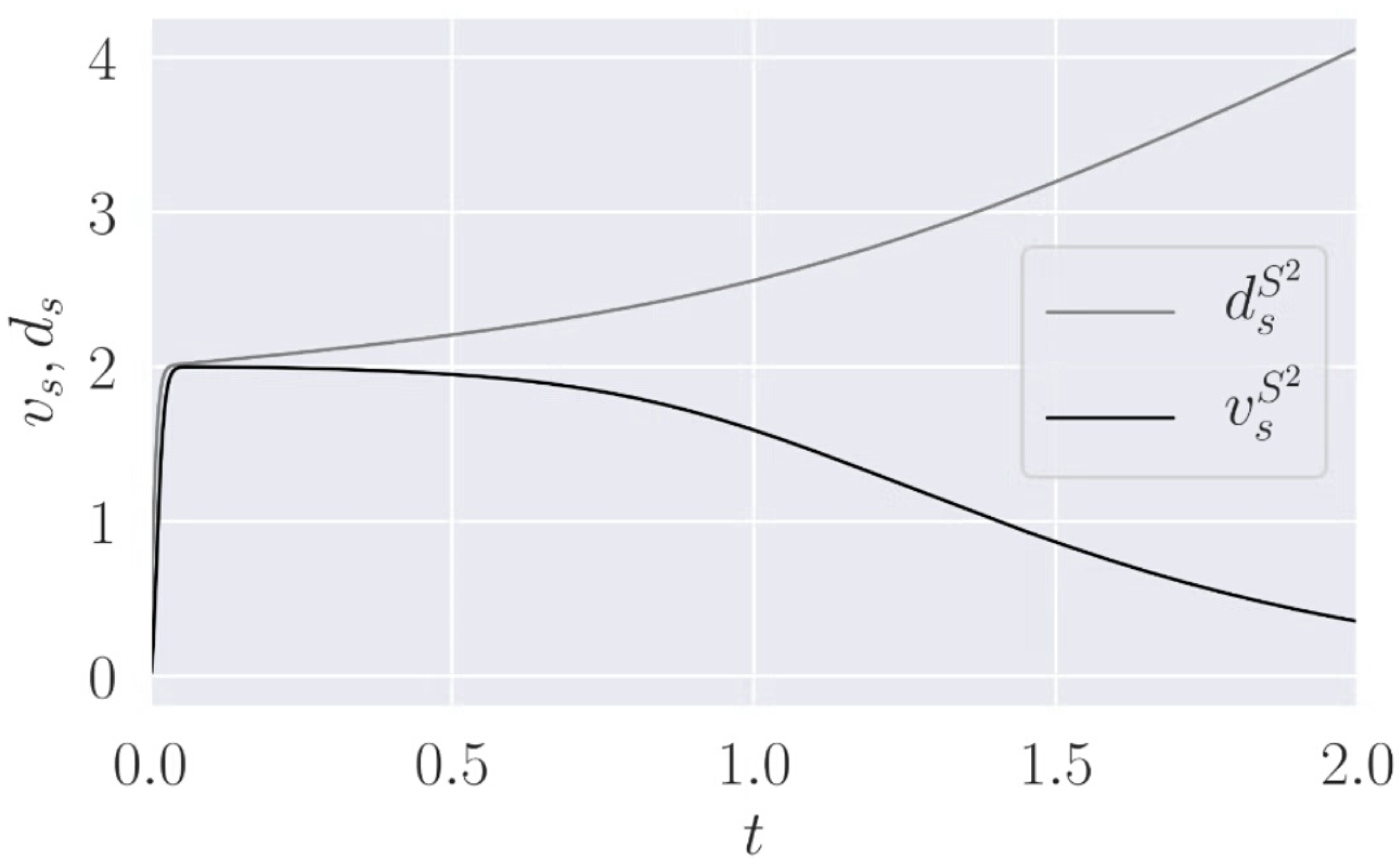
$$K(t) = \sum_{\lambda} e^{-t\lambda^2}$$

$$d_s(t) = -2\,\frac{d\log K}{d\log t}, \qquad V_s(t) = d_s - t\,\frac{d(d_s)}{dt}$$

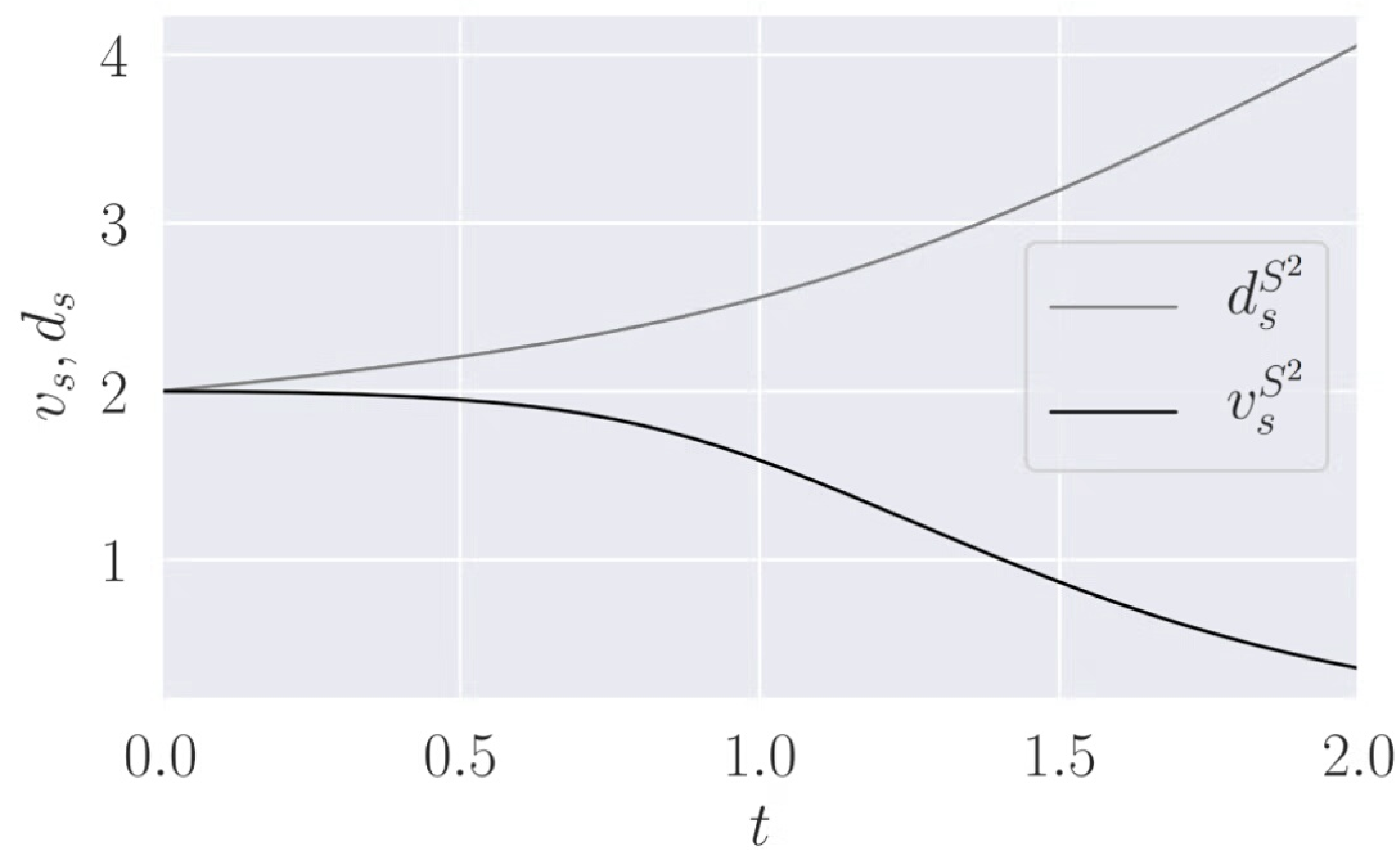
$$= 2t^2\left(\frac{\sum \lambda^4 e^{-t\lambda^2}}{\sum e^{-t\lambda^2}} - \left(\frac{\sum \lambda^2 e^{-t\lambda^2}}{\sum e^{-t\lambda^2}}\right)^2\right)$$

$= 2 \times$ heat capacity

Spectral dimension & variance, S^2



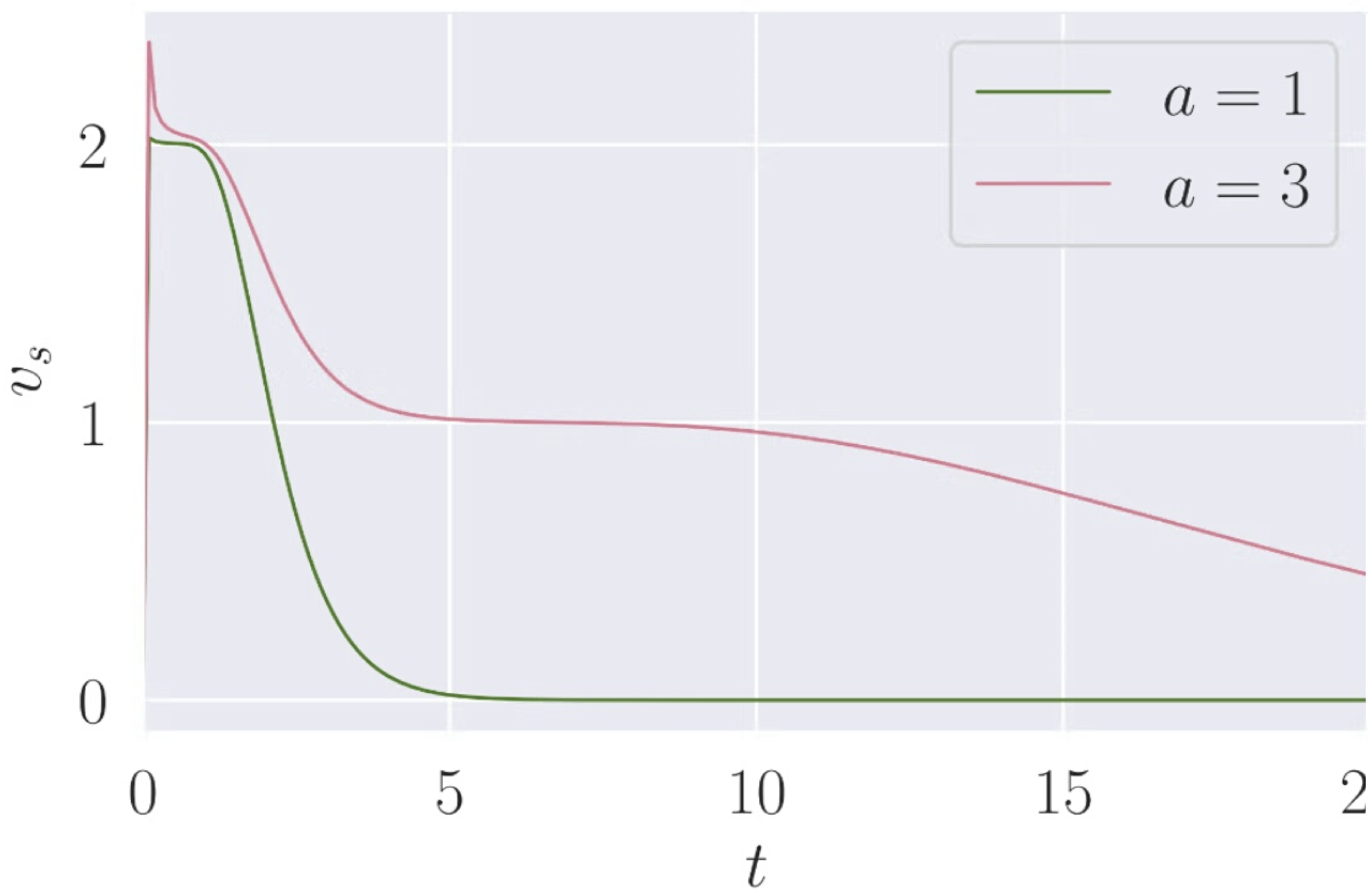
N = 15



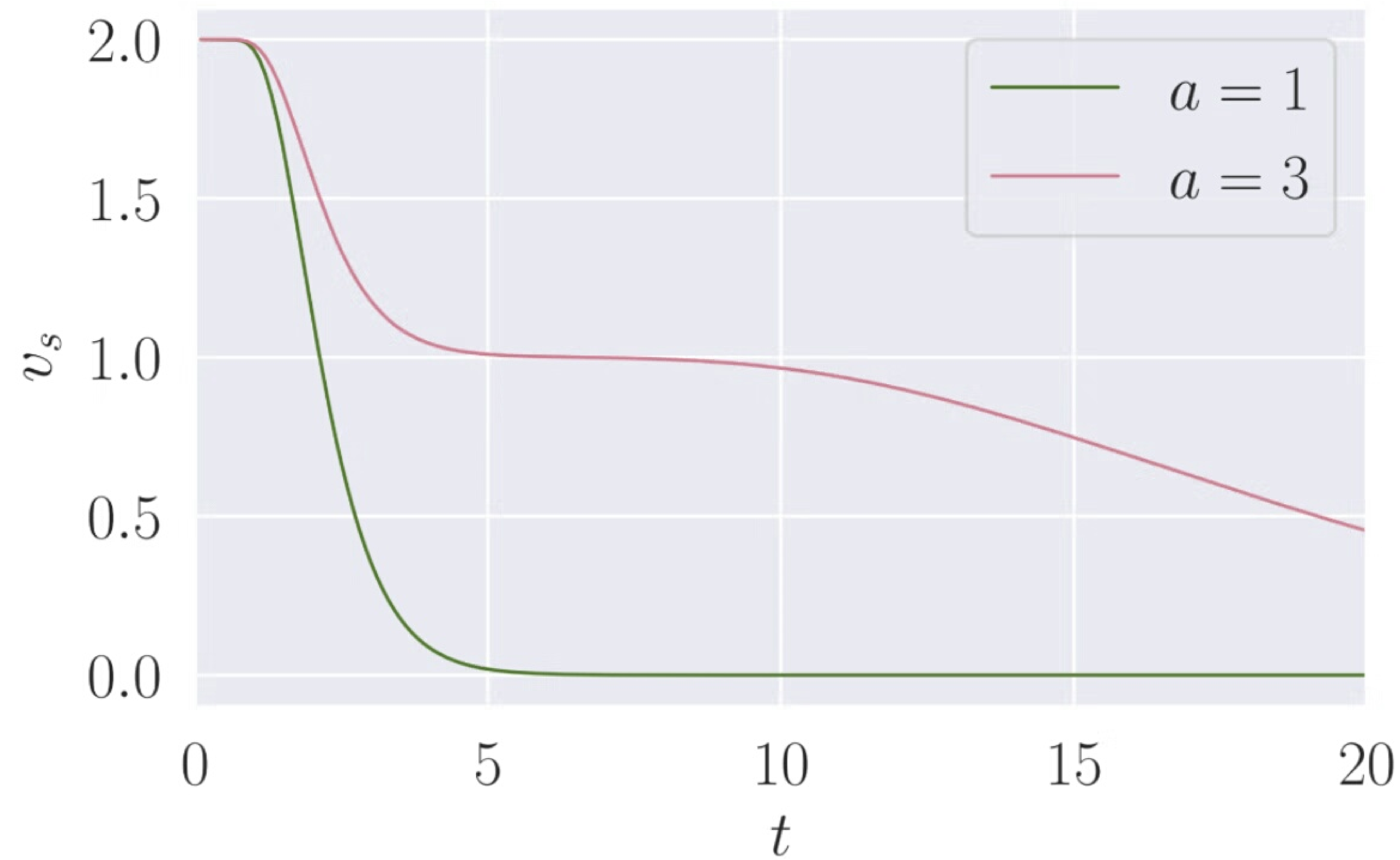
Continuum

Spectral variance, torus

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$



$N = 90$



Continuum

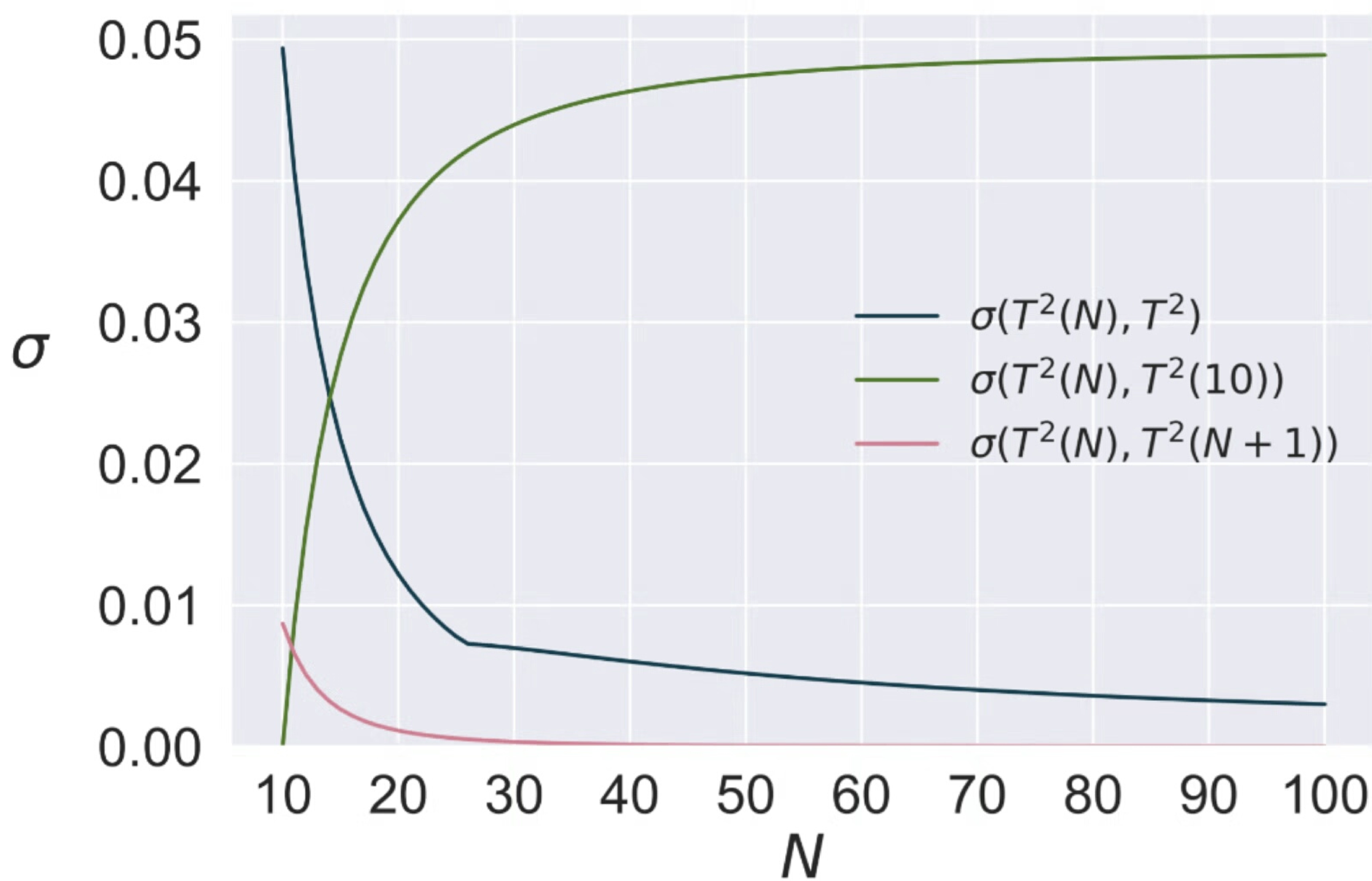
Distance between spectra

Spectral zeta: $\zeta(s) = \sum (\lambda^2)^{-s}$

$$\sigma(D_1, D_2) = \sup_{r \leq s \leq r+1} \left| \log \frac{\zeta_1(s)}{\zeta_2(s)} \right|$$

Cornelissen, Kontogeorgis

Distance between square tori



Random geometries

$$\mathcal{G} = \{ D \text{ satisfying axioms} \} \qquad \text{Vector space}$$

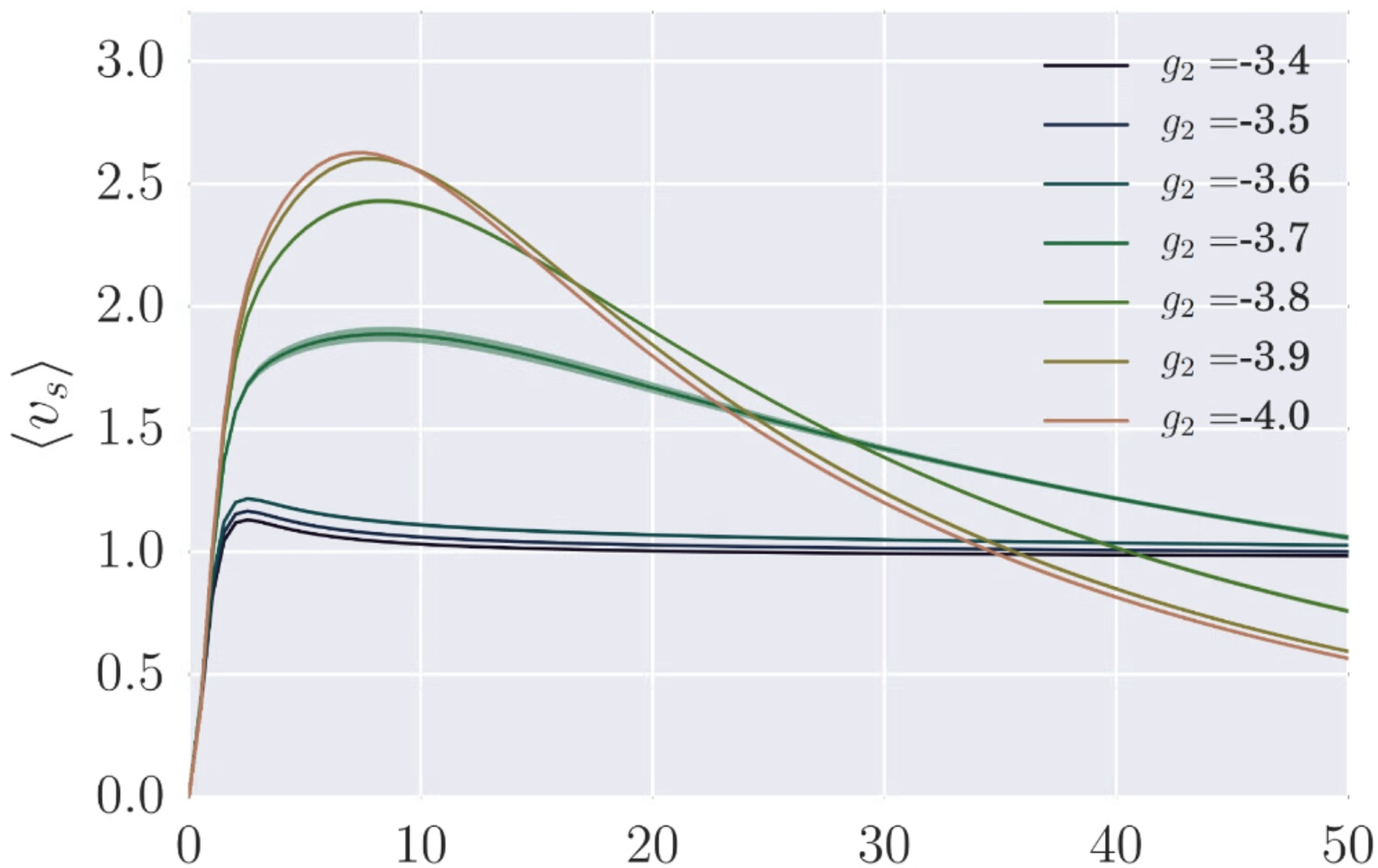
$$\langle f \rangle = \frac{\int_{\mathcal{G}} f(D)\, e^{-\operatorname{tr}(D^4 + g_2 D^2)}\, dD}{\int_{\mathcal{G}} e^{-\operatorname{tr}(D^4 + g_2 D^2)}\, dD}$$

Type (1,3)

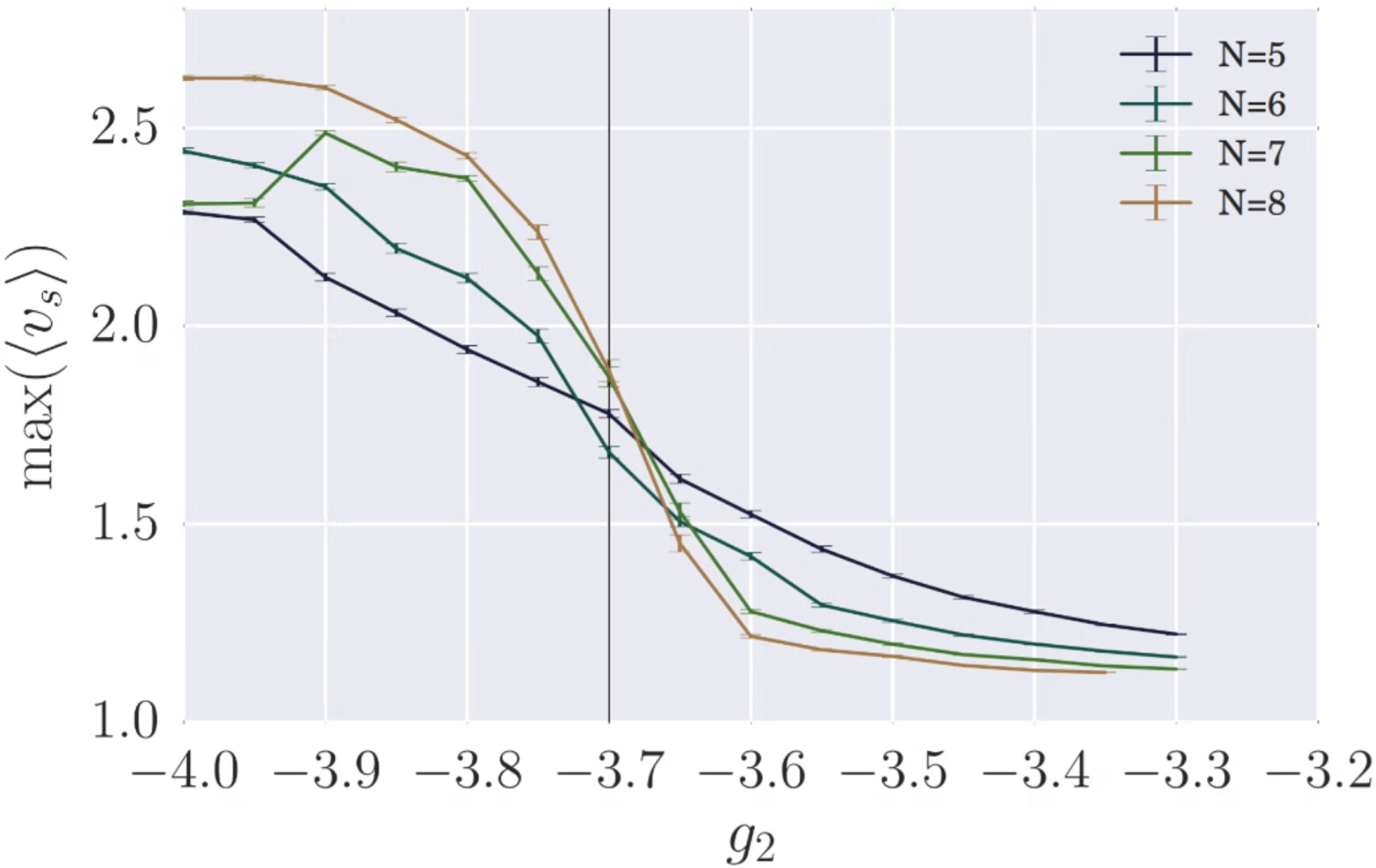
$$
\begin{aligned}
D\psi = {} & \gamma^0\{H_0,\psi\} + \gamma^1[L_1,\psi] + \gamma^2[L_2,\psi] \\
& + \gamma^3[L_3,\psi] + \gamma^0\gamma^1\gamma^2[L_{012},\psi] \\
& + \gamma^0\gamma^2\gamma^3[L_{023},\psi] + \gamma^0\gamma^3\gamma^1[L_{031},\psi] \\
& + \gamma^1\gamma^2\gamma^3\{H_{123},\psi\}
\end{aligned}
$$

H, L random

Spectral variance, random (1,3)



Maximum of the spectral variance



Distance $\sigma(S^2, \text{random}(1,3))$