Global Seiberg-Witten quantization for U(n)-bundles on tori

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Based on arXiv:1809.05426 with Paolo Aschieri[2,3]

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11-15 February 2019
Quantum Spacetime '19, Bratislava

Context and motivation

- Well-studied area in mathematics $\cap$ physics: Yang-Mills on noncommutative tori (Connes, Rieffel, 1987).
- Relation to M-theory (Connes, Douglas, Schwarz, 1998).
- Study of modules over noncommutative tori, corresponcence Heisenberg-modules to quantized $U(n)$-bundles, SYM over Morita equivalent torus algebras have the same BPS spectrum (Ho, Konechny, Schwarz, 1998).
- Open string sector: Gauge theory on the worldvolume of D-branes with backgound Kalb-Ramond field is noncommutative. Relation between commutative and noncommutative theory: Local definition of Seiberg-Witten (SW) map (Seiberg, Witten, 1999).
- Kontsevich formality theorem used to quantize line bundles over arbitrary Poisson manifolds with SW maps (Jurčo, Schupp, Wess, 2002).
- In this talk: Define SW map for $U(n)$-bundles over tori and study compatibility with Morita equivalence/ T-duality.

The Seiberg-Witten (SW) map

- M k-dim. manifold, $\theta \in \Gamma(\wedge^2 T^* M)$ Poisson structure.
- $E \to M$ n-dim. vector bundle, associated to a principal G-bundle with local connection form $A \in \Omega^1(M, \mathfrak{g})$.
- $\mathcal{E} \in \mathcal{M}_{\mathcal{C}^\infty(M)}$ $\mathcal{C}^\infty(M)$-module of sections in E, $U \subset M$, $\mathcal{E}|_U \simeq \mathcal{C}^\infty(\mathbb{R}^k)^{\oplus n}$.
- $\star$: Moyal-Weyl star product on $\mathbb{R}^k_\theta$ if θ constant, $\theta|_U$ seen as $\theta \in \wedge^2 T^* \mathbb{R}^k$.

Definition (Seiberg, Witten, 1999)

The SW map locally relates a commutative gauge theory $(\mathcal{E}|_U, A|_U, \cdot, \theta|_U)$ to a noncommutative gauge theory $(\hat{\mathcal{E}} \simeq \mathcal{C}^\infty(\mathbb{R}^k_\theta)^{\oplus n}, \hat{A}, \star)$ such that commutative gauge variations are mapped into noncommutative ones:

$$\hat{\Phi}(A + \delta_\varepsilon A, \Phi + \delta_\varepsilon \Phi) = \hat{\Phi}(A, \Phi) + \hat{\delta}_{\hat{\varepsilon}} \hat{\Phi}(A, \Phi) \, . \qquad (1)$$

where Φ,$\hat{\Phi}$ stand for the connection or sections.

e.g. for the connection A itself: $\hat{A}(A + \delta_\varepsilon A) = \hat{A}(A) + \hat{\delta}_{\hat{\varepsilon}} \hat{A}(A)$, where

$$\begin{aligned} \delta_\epsilon A_\mu &= \partial_\mu \varepsilon - i A_\mu \varepsilon + i \varepsilon A_\mu \\ \hat{\delta}_{\hat{\varepsilon}} \hat{A}_\mu &= \partial_\mu \hat{\varepsilon} - i \hat{A}_\mu \star \hat{\varepsilon} + i \hat{\varepsilon} \star \hat{A}_\mu \, . \end{aligned} \qquad (2)$$

The Seiberg-Witten (SW) map

Expanding the SW condition for $\theta^{\mu\nu} \to \theta^{\mu\nu} + \delta\theta^{\mu\nu}$ gives the SW differential equations for connection $\hat{A}(A)$, gauge parameter $\hat{\varepsilon}(\varepsilon, A)$ and sections $\hat{\phi}(\phi, A)$, e.g. in the fundamental rep.:

$$\delta\theta^{\mu\nu}\frac{\partial \hat{A}_\kappa}{\partial\theta^{\mu\nu}} = \frac{\pi}{2}\delta\theta^{\mu\nu}\Big(\hat{A}_\mu \star (\partial_\nu \hat{A}_\kappa + \hat{F}_{\nu\kappa}) + (\partial_\nu \hat{A}_\kappa + \hat{F}_{\nu\kappa}) \star \hat{A}_\mu\Big) \ . \tag{3}$$

$$\delta\theta^{\mu\nu}\frac{\partial \hat{\varepsilon}}{\partial\theta^{\mu\nu}} = \frac{\pi}{2}\delta\theta^{\mu\nu}\Big(\partial_\mu \hat{\varepsilon} \star \hat{A}_\nu + \hat{A}_\nu \star \partial_\mu \hat{\varepsilon}\Big) \ . \tag{4}$$

$$\delta\theta^{\mu\nu}\frac{\partial \hat{\phi}}{\partial\theta^{\mu\nu}} = -\frac{\pi}{2}\delta\theta^{\mu\nu}\Big(\hat{A}_\mu \star \partial_\nu \hat{\phi} + \hat{A}_\mu \star D_\nu \hat{\phi}\Big) \ . \tag{5}$$

- $\hat{F}_{\mu\nu} := \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu \star \hat{A}_\nu + i\hat{A}_\nu \star \hat{A}_\mu$.
- Similar differential equations for fields in adjoint rep.
- Explicit recursive solutions: Formal power series in θ.

Question: Seiberg Witten maps globally for non-trivial bundles?

- Line bundles (Jurčo, Schupp, Wess, 2002)
- Now: $U(n)$-bundles over tori

Noncommutative $U(n)$-bundles via twisted boundary conditions

- $T \simeq \mathbb{R}^2_{\sigma^1,\sigma^2}/(2\pi\mathbb{Z})^2$, 2π-periodic functions $\mathcal{C}^\infty(T) \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^2)$.
- Module of sections of $U(n)$-bundle over T: Take trivial $\mathcal{C}^\infty(\mathbb{R}^2)^{\oplus n}$ together with $U(n)$-matrix valued functions $\Omega_1(\sigma^2), \Omega_2(\sigma^1)$ satisfying a cocycle condition and determining twisted boundary conditions.
- This gives $\mathcal{C}^\infty(T)$-module $\mathcal{E}_{n,m}$, carrying a connection A with topological charge m.

- NC torus: $\mathcal{C}^\infty(T_\theta) =: T_\theta \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^2_\theta)$, $e^{i\sigma^1} \star e^{i\sigma^2} = e^{-2\pi i\theta}\, e^{i\sigma^2} \star e^{i\sigma^1}$.
- Module of sections on $U(n)$-bundle over T_θ: Subset of $\mathcal{C}^\infty(\mathbb{R}^2_\theta)^{\oplus n}$, s.t.

$$\phi^\theta(\sigma^1 + 2\pi, \sigma^2) = \Omega_1(\sigma^2) \star \phi^\theta(\sigma^1, \sigma^2)\ , \tag{6}$$

$$\phi^\theta(\sigma^1, \sigma^2 + 2\pi) = \Omega_2(\sigma^1) \star \phi^\theta(\sigma^1, \sigma^2)\ , \tag{7}$$

$$\Omega^1(\sigma^2 + 2\pi) \star \Omega_2(\sigma^1) = \Omega_2(\sigma^1 + 2\pi) \star \Omega_1(\sigma^2)\ . \tag{8}$$

- Connection, e.g. $D_1 = \partial_{\sigma^1}, D_2 = \partial_{\sigma^2} - \frac{i}{2\pi}\frac{m\sigma^1}{n-m\theta}\, 1_{n\times n}$.
- As bimodule, we write $\mathcal{E}^\theta_{n,m} \in {}_{\mathrm{End}(\mathcal{E}^\theta_{n,m})}\mathcal{M}_{T_{(-\theta)}}$.

The induced SW map: From $\mathbb{R}^2$ to the torus

SW map: Local $\leftrightarrow$ for trivial bundles/bimodules:

$$(E = \mathcal{C}^\infty(\mathbb{R}^2)^{\oplus n} \in {}_{\mathrm{End}(E)}\mathcal{M}_{\mathcal{C}^\infty(\mathbb{R}^2)}, A_\mu, \theta) \xrightarrow{\text{SW map}} (\hat{E} = \mathcal{C}^\infty(\mathbb{R}^2_\theta)^{\oplus n} \in {}_{\mathrm{End}(\hat{E})}\mathcal{M}_{\mathcal{C}^\infty(\mathbb{R}^2_\theta)}, \hat{A}_\mu) \quad (9)$$

Idea to induce a SW map for bundles on the torus:

- For $(\mathcal{E} \in {}_{\mathrm{End}(\mathcal{E})}\mathcal{M}_{\mathcal{C}^\infty(T)}, A_\mu)$, see $\mathcal{E}$ as a linear subspace of $E \in {}_{\mathrm{End}(E)}\mathcal{M}_{\mathcal{C}^\infty(\mathbb{R}^2)}$ and as bimodule with respect to $\mathrm{End}(\mathcal{E}) \hookrightarrow \mathrm{End}(E)$ and $\mathcal{C}^\infty(T) \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^2)$.
- Apply the SW map to get $(\hat{\mathcal{E}}, \hat{A}_\mu)$ with $\hat{\mathcal{E}}$ as a linear subspace of $\hat{E} \in {}_{\mathrm{End}(\hat{E})}\mathcal{M}_{\mathcal{C}^\infty(\mathbb{R}^2_\theta)}$.
- Prove: The noncommutative twisted boundary conditions are satisfied for this subspace, similarly for the endomorphism algebra of this subspace.

Definition

$(\hat{\mathcal{E}}, \hat{A}_\mu)$ is called the *SW quantization of* $(\mathcal{E}, A_\mu)$, where $\hat{\mathcal{E}}$ is the subset of all elements in $\hat{E} = \mathcal{C}^\infty(\mathbb{R}^2_\theta)^{\oplus n}$ that satisfy the noncommutative twisted boundary conditions with the SW quantized $\hat{\Omega}_1(\sigma^2), \hat{\Omega}_2(\sigma^1)$.

Result: Induced SW map for $U(n)$-bundles on the torus

Theorem (Aschieri, Deser, 2018)

The induced SW map on torus bundles $(\mathcal{E}_{n,m}, A_\mu) \xrightarrow{\text{SW induced}} (\hat{\mathcal{E}}_{n,m}, \hat{A}_\mu)$ satisfies

- Let $\phi \in E$ satisfy the commutative twisted boundary conditions, then its SW quantization $\hat{\phi}$ satisfies the noncommutative twisted boundary conditions, i.e. $\hat{\phi} \in \mathcal{E}^\theta_{n,m}$.
- Let $\Psi \in End(E)$ satisfy the commutative twisted boundary conditions (adjoint), then its SW quantization $\hat{\Psi}$ satisfies the noncommutative twisted boundary conditions, i.e. $\hat{\Psi} \in End(\mathcal{E}^\theta_{n,m})$.

Consequently, we have the commutative diagram

$$\begin{array}{ccc}
(E \in {}_{\mathrm{End}(E)}\mathcal{M}_{\mathcal{C}^\infty(\mathbb{R}^2)}, A_\mu, \theta) & \xrightarrow{\text{SW map}} & (\hat{E} = E^\theta \in {}_{\mathrm{End}(E^\theta)}\mathcal{M}_{\mathcal{C}^\infty(\mathbb{R}^2_\theta)}, \hat{A}_\mu) \\
\uparrow i & & \uparrow i_\theta \\
(\mathcal{E}_{n,m} \in {}_{\mathrm{End}(\mathcal{E}_{n,m})}\mathcal{M}_{\mathcal{C}^\infty(T)}, A_\mu, \theta) & \xrightarrow{\text{SW induced}} & (\hat{\mathcal{E}}_{n,m} = \mathcal{E}^\theta_{n,m} \in {}_{\mathrm{End}(\mathcal{E}^\theta_{n,m})}\mathcal{M}_{T_{(-\theta)}}, \hat{A}_\mu)
\end{array}$$

Sketch: SW map vs T-duality/Morita equivalence

The iduced SW map enables to study the relation of T-duality (modules over Morita equivalent tori) and SW quantization.

Definition: Gauge Morita equivalence

Two torus algebras $(A_\theta, \tilde{A}_{\tilde{\theta}})$ are *gauge Morita equivalent* if there exists a Morita equivalence bimodule $P \in {}_{A_\theta}\mathcal{M}_{\tilde{A}_{\tilde{\theta}}}$ equipped with a constant curvature bimodule connection.

- P relates right A-modules with A-connections to right $\hat{A}$-modules with $\hat{A}$-connections via the tensor product over A.
- Specifying to $\mathcal{E}^\theta_{n,m} \in \mathcal{M}_{T_{(-\theta)}}$ and a (gauge) Morita equivalence bimodule $P \in {}_{T_{(-\theta)}}\mathcal{M}_{T_{(-\tilde{\theta})}}$ we have a duality transformation

$$\mathcal{E}^\theta_{n,m} \in \mathcal{M}_{T_{(-\theta)}} \to \mathcal{E}^\theta_{n,m} \otimes_{T_{(-\theta)}} P \simeq \mathcal{E}^{\tilde{\theta}}_{\tilde{n},\tilde{m}} \in \mathcal{M}_{T_{(-\tilde{\theta})}} . \tag{10}$$

- Result (Connes, Douglas, Morariu, Schwarz, Zumino), that (super-) Yang-Mills theories for the modules $\mathcal{E}^\theta_{n,m}$ and $\mathcal{E}^{\tilde{\theta}}_{\tilde{n},\tilde{m}}$ have the same BPS spectrum.

Sketch: SW map vs T-duality/Morita equivalence

- For $\mathcal{E}^{\theta}_{n,m}$, one can show that the SW map does not change (n, m), so $(\mathcal{E}^{\theta}_{n,m}, A^{\theta}_{\mu})$ in general is mapped to $(\mathcal{E}^{\theta'}_{n,m}, A^{\theta'}_{\mu})$.
- Natural to investigate SW maps before and after applying the duality transformation $\otimes_{T_{(-\theta)}} P$. There is a compatibility:

Compatibility of SW and duality (Aschieri, Deser, 2018)

For $\mathcal{E}^{\theta}_{n,m} \in \mathcal{M}_{T_{(-\theta)}}$ which is mapped via SW to $\mathcal{E}^{\theta'}_{n,m} \in \mathcal{M}_{T_{(-\theta')}}$ and gauge Morita equivalence bimodules $P \in {}_{T_{(-\theta)}}\mathcal{M}_{T_{(-\tilde{\theta})}}$ and $P' \in {}_{T_{(-\theta')}}\mathcal{M}_{T_{(-\tilde{\theta}')}}$ we have the commutative diagram

$$\begin{array}{ccc} (\mathcal{E}^{\theta}_{n,m}, A^{\theta}_{\mu}) & \xrightarrow{\otimes_{T_{(-\theta)}} P} & (\mathcal{E}^{\tilde{\theta}}_{\tilde{n},\tilde{m}}, A^{\tilde{\theta}}_{\mu}) \\ {\scriptstyle SW^{\theta'}_{\theta}}\downarrow & & \downarrow{\scriptstyle SW^{\tilde{\theta}'}_{\tilde{\theta}}} \\ (\mathcal{E}^{\theta'}_{n,m}, A^{\theta'}_{\mu}) & \xrightarrow{\otimes_{T_{(-\theta')}} P'} & (\mathcal{E}^{\tilde{\theta}'}_{\tilde{n},\tilde{m}}, A^{\tilde{\theta}'}_{\mu}) \end{array} \tag{11}$$

Further remarks, outlook

- Known fact: SW maps are ambiguous. We wrote the ambiguities in a transparent way and used them to show that explicit solutions for sections in $\mathcal{E}^{\theta}_{n,m}$ known in the literature (Ho, 1998) can be obtained with SW quantization of $\mathcal{E}_{n,m}$.
- The Theorem on the induced SW map remains true if one includes these ambiguities.

- Induced SW maps for bundles over higher dimensional tori (used in studies of T-duality in string theory), and more general for bundles over quotients of $\mathbb{R}^k$.
- Physics meaning of compatibility of SW with duality transformations?
- Applications to closed string theory? Heisenberg nilfolds etc.

Appendix on SW map: Ambiguities

Well known fact: SW differential equations are ambiguous (Asakawa, Kishimoto, 1999). We approach this by adding extra terms $\hat{D}_{\mu\nu\kappa}(\hat{A})$, $\hat{E}_{\mu\nu}(\hat{\varepsilon},\hat{A})$ and $C_{\mu\nu}(\hat{\phi},\hat{A})$:

$$\delta\theta^{\mu\nu}\frac{\partial \hat{A}_\kappa}{\partial\theta^{\mu\nu}} = \frac{\pi}{2}\delta\theta^{\mu\nu}\Big(\hat{A}_\mu\star(\partial_\nu\hat{A}_\kappa+\hat{F}_{\nu\kappa})+(\partial_\nu\hat{A}_\kappa+\hat{F}_{\nu\kappa})\star\hat{A}_\mu + D_{\mu\nu\kappa}(\hat{A})\Big)\ .$$
$$\delta\theta^{\mu\nu}\frac{\partial \hat{\varepsilon}}{\partial\theta^{\mu\nu}} = \frac{\pi}{2}\delta\theta^{\mu\nu}\Big(\partial_\mu\hat{\varepsilon}\star\hat{A}_\nu+\hat{A}_\nu\star\partial_\mu\hat{\varepsilon}+E_{\mu\nu}(\hat{\varepsilon},\hat{A})\Big)\ .$$
$$\delta\theta^{\mu\nu}\frac{\partial \hat{\phi}}{\partial\theta^{\mu\nu}} = -\frac{\pi}{2}\delta\theta^{\mu\nu}\Big(\hat{A}_\mu\star\partial_\nu\hat{\phi}+\hat{A}_\mu\star D_\nu\hat{\phi}+C_{\mu\nu}(\hat{\phi},\hat{A})\Big)\ . \tag{12}$$

SW condition is satisfied if

$$\hat{D}_{\mu\nu\kappa}(\hat{A}+\hat{\delta}_{\hat{\varepsilon}}\hat{A})-\hat{D}_{\mu\nu\kappa}(\hat{A})-i[\hat{\varepsilon},\hat{D}_{\mu\nu\kappa}(\hat{A})]_\star = -D_\kappa\hat{E}_{\mu\nu}(\hat{\varepsilon},\hat{A})\ .$$
$$\hat{C}_{\mu\nu}(\hat{A}+\hat{\delta}_{\hat{\varepsilon}}\hat{A},\hat{\phi}+\hat{\delta}_{\hat{\varepsilon}}\hat{\phi})-\hat{C}_{\mu\nu}(\hat{A},\hat{\phi})-i\hat{\varepsilon}\star\hat{C}_{\mu\nu}(\hat{A},\hat{\phi}) = -i\hat{E}_{\mu\nu}(\hat{\varepsilon},\hat{A})\star\hat{\phi}\ .$$

- Similar for fields in the adjoint representation.
- In case $\hat{E}_{\mu\nu}=0$, any $\hat{D}_{\mu\nu\kappa}$ and $\hat{C}_{\mu\nu}$ covariant under gauge transformations are solutions to these conditions.

Appendix: Ho's quantum $U(n)$-bundle via SW

There's a known solution to the noncommutative twisted boundary conditions (HO,1998).

$$\phi_k^\theta(\sigma^1,\sigma^2) = \sum_{s\in\mathbb{Z}}\sum_{j=1}^{m} E\left(\tfrac{m}{n}(\tfrac{\sigma^2}{2\pi}+k+ns)+j, i\sigma^1\right)\star\tilde{\phi}_j(\tfrac{\sigma^2}{2\pi}+k+ns+\tfrac{n}{m}j)\ , \quad (13)$$

where $\tilde{\phi}_j(x)$ are Schwartz functions on $\mathbb{R}\times\mathbb{Z}_m$ and $E(A,B)$ is a normal ordered version of the exponential function $E(A,B) := \frac{1}{1-[A,B]_\star}\sum_{k=0}^{\infty}\frac{1}{k!}A^k\star B^k$.

We have the following

Observation

The function $\phi^\theta = (\phi_k^\theta)_{k=1\dots n}$ satisfies the differential equation

$$\frac{\partial}{\partial\theta}\phi^\theta - \pi\,\hat{A}_2\star\partial_1\phi^\theta = 3\pi\,\hat{F}\star\phi^\theta + i\pi\,D_1D_2\phi^\theta\ . \quad (14)$$

Hence, taking $\hat{C}_{12} = -\hat{C}_{21} = -3\hat{F}\star\phi^\theta - iD_1D_2\phi^\theta$ shows, that ϕ^θ satisfies the SW equation for fundamental sections.