EMERGENT GEOMETRIES

from Superstring Field Theory.

P.A. Grassi, Università del Piemonte Orientale, Italy

in collaboration with
Roberto Catenacci (Univ. Piemonte Orientale) and Simone Noja (Univ. Insubria)

based on arXiv:1807.09563, arXiv:1901.00818

contributions also in collaborations with
Leonardo Castellani (UPO) and Pietro Fré (UNITO)

- Chern-Simons theory on a 3-manifold and the similarities with the bosonic string field theory (SFT).

- Integration on supermanifolds, integral, pseudo and differential forms, the PCO's and the supersymmetric actions.

- Super-Chern-Simons theory on a (3|2)-supermanifold and the super SFT.

- Emergent geometric structures, the interactions terms for super Chern-Simons theories and the non-associative algebras.

- Outlook and perspectives.

Chern-Simons theory and SFT

Chern-Simon theory on a 3-manifold

The topological gauge theory of Chern-Simons is described by the functional

$$S_{CS} = \int_{\mathcal{M}^3} \mathrm{Tr}\Big(A^{(1)} \wedge dA^{(1)} + \frac{2}{3} A^{(1)} \wedge A^{(1)} \wedge A^{(1)}\Big)$$

is built in terms of a gauge connection $A^{(1)}$

with values in the Lie algebra of a gauge group $\mathcal{G}$

Products of the gauge fields correspond to the wedge products for 3d differential forms and as matrix multiplication for Lie-algebra valued fields.

The integral is over the 3d manifold and the theory is able to measure topological characteristics of the manifold, but not the metric structures.

The EoM's correspond to the vanishing curvature of the connection

$$F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)} = 0$$

CS theory and bosonic SFT share several similarities

$$S_{SFT} = \left\langle \Phi^{(1)}, Q\Phi^{(1)} \right\rangle + \frac{2}{3} \left\langle \Phi^{(1)}, \Phi^{(1)} \star \Phi^{(1)} \right\rangle$$

The gauge connection is replaced by the string field $\Phi^{(1)}$

The differential is replaced by the BRST operator Q

The form degree is replaced the ghost number, the product between forms is replaced by the complicate Witten star product (also defined in terms of the underlying conformal field theory)

The bilinear form < , > is a cyclic invariant product (sometimes expressed as an integral) and replaces the integral and the Trace for gauge fields.

For topological strings the BSFT action reduces to CS theory for a Lagrangian submanifold.

In the case of CS theory, there is no propagating d.o.f. and one can compute non local observables such as the Wilson loops

$$\mathcal{W}(\gamma) = \mathrm{Tr}\mathcal{P}e^{i\oint_\gamma A^{(1)}}$$

For SFT, there are infinite propagating d.o.f.'s and it represents off-shell realisation of first quantised string theory. There are several observables (not discussed in the present talk).

BV-BRST formalism

For CS theory, the BV-BRST formalism can be constructed by considering a generalised form (one needs to separate between fields and antifields)

$$\mathcal{A} = A^{(1)} + A^{(0)} + A^{(-1)} + A^{(-2)} \equiv C + A + A^* + C^*$$

Analogoulsy for SFT, but with infinite components (in analogy with supermanifold pseudo-forms which I will discuss next)

$$\Phi = \sum_{p=-\infty}^{\infty} \Phi^{(p)} = \sum_{p=-\infty}^{0} \Phi^{(p)} + \sum_{p=1}^{\infty} \Phi^{(p)} \equiv \text{fields} \ + \ \text{antifields}$$

Integration on supermanifolds

Integration of Forms on Supermanifolds

Let us begin with a conventional manifold $\mathcal{M}$ with dimension = n, given a generic differential form

$$\omega \in \Omega^{\bullet}(\mathcal{M})$$

This is a section of the exterior bundle and it can be decomposed as

$$\omega = \omega^0 + \omega^1 + \omega^2 + \cdots + \omega^n$$

where the last term is the top form. Locally, a generic form can be written as

$$\omega(x, dx) = \sum_{p=0}^{n} \omega_{[\mu_1 \ldots \mu_p]}(x) dx^{\mu_1} \ldots dx^{\mu_p}$$

and its integral on the manifold is

$$\int_{\mathcal{M}} \omega = \int f(x)[d^n x] \,, \quad f(x) = \sqrt{g}\, \omega_{[\mu_1 \ldots \mu_n]}(x) \epsilon^{\mu_1 \ldots \mu_n}$$

where the second member is a Lebesgue/Riemann integral of the function built in terms of the differential form.

Differential forms on a supermanifold

Let us now move to supermanifolds. We denote by $\mathcal{M}$ a (n|m)-dimensional supermanifold parametrised by the local coordinates (x^μ, θ^α)

We introduce also the corresponding 1-forms $(dx^\mu, d\theta^\alpha)$ with the properties

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \quad dx^\mu \wedge d\theta^\alpha = d\theta^\alpha \wedge dx^\mu \quad d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha$$

Then a generic (super) form looks like

$$\omega = \sum_{k=1,l=1}^{k=p,l=q} \omega_{[\mu_1 \dots \mu_k](\alpha_1 \dots \alpha_l)} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} d\theta^{\alpha_1} \wedge \dots \wedge d\theta^{\alpha_l}$$

where the components $\omega_{[\mu_1 \dots \mu_k](\alpha_1 \dots \alpha_l)}(x,\theta)$ are functions of the manifold coordinates. The indices $[\mu_1 \dots \mu_k]$ are antisymmetrized while $(\alpha_1 \dots \alpha_l)$ are symmetrized. The total form degree is fixed by the p + q, summing the form degree of the bosonic coordinates and the form degree of the fermionic ones.

This implies that there is no upper bound to the form degree and there is no top form.

The integrals over the fermionic coordinates (dx, θ) are Berezin integrals, over the x-coordinates are the usual Lebesgue/Riemann integrals, but the integral over dθ is not well defined on the superforms.

To define the integration over dθ we need a new quantity $\delta(d\theta^\alpha)$

such that $$\int f(d\theta^\alpha)\delta(d\theta^\alpha) = f(0)$$

with the usual properties $$d\delta(d\theta^\alpha) = \delta'(d\theta)d^2\theta = 0$$

They formally share all distributional properties of the usual Dirac delta functions. In addition, they are forms and therefore we can apply the usual geometric differential operators.

For the Dirac delta functions we assume the following properties (distributional properties)

$$\delta(d\theta^\alpha) \wedge \delta(d\theta^\beta) = -\delta(d\theta^\beta) \wedge \delta(d\theta^\alpha)$$

this follows by assuming an oriented integration measure. In this way, we see that there is an upper bound to the number of delta's: the number of fermionic coordinates.

A fundamental property is the distributional equation

$$d\theta^\alpha \delta(d\theta^\alpha) = 0$$

In the same way, using the distributional properties of delta's, we have that

$$d\theta^\alpha \delta^{(n)}(d\theta^\alpha) = -n\delta^{(n-1)}(d\theta^\alpha)$$

That equation tells us that the derivatives of delta's carry a negative form degree. In this way, multiplying by dθ, it reduces the negative power. The Dirac delta has no form degree.

Now a generic **(pseudo)-form** can be written as

$$\omega = \sum_{p,r,s} \omega_{[\mu_1 \dots \mu_p](\alpha_1 \dots \alpha_r)[\alpha_{r+1} \dots \alpha_s]}(x,\theta) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \wedge d\theta^{\alpha_1} \wedge \dots \wedge d\theta^{\alpha_r} \wedge \delta(d\theta^{\alpha_{r+1}}) \wedge \dots \wedge \delta(d\theta^{\alpha_s})$$

each pieces are differential forms with fixed **form degree = p + r** and **picture number = s - r**

A generic (p|q) form is written in terms of

$$dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} d\theta^{\alpha_1} \wedge \dots \wedge d\theta^{\alpha_r} \delta^{(s_1)}(d\theta^{\beta_1}) \wedge \dots \delta^{(s_q)}(d\theta^{\beta_q})$$

and we denote by $\Omega^{(p|q)}(\mathcal{M})$ the space of pseudo-forms. For q=0, we have the well-known **superforms**, for q=m we have the **integral forms** and for 0< q<m, we have the space of **pseudo-forms**.

We can apply the complete Cartan calculus (Lie derivatives, contractions, inner products….)

Form complexes

Now we have the following complexes

$$0 \to \Omega^{(0|0)} \to \Omega^{(1|0)} \to \dots \to \Omega^{(n|0)} \to \Omega^{(n+1|0)} \dots$$

where all spaces are finite dimensional. The complex is not bounded from above.
The differential **d** acts along the arrows.

$$\dots \to \Omega^{(-2|m)} \to \Omega^{(-1|m)} \to \dots \to \boxed{\Omega^{(n|m)}} \to 0$$

this is the complex of integral forms. It is unbounded from below, but it is bounded from above.
The last space is the space of top forms. Notice that when we have the maximum number of delta's, there is no room for $d\theta' s$

There are additional complexes of the form:

$$\dots \to \Omega^{(-2|q)} \to \Omega^{(-1|q)} \to \dots \to \Omega^{(n|q)} \to \dots$$

which is not bounded from above nor from below. In addition, each single space is **infinite dimensional** space and their geometry is completely unknown .

In summary, we have

$$\begin{array}{ccccccccccccc}
 & 0 & \xrightarrow{\mathrm{d}} & \Omega^{(0|0)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|0)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|0)} & \xrightarrow{\mathrm{d}} & \Omega^{(m+1|0)} & \dots \\
 & Z\uparrow & & Z\uparrow\downarrow Y & & & Z\uparrow\downarrow Y & & & Z\uparrow\downarrow Y & & \downarrow Y & \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots & \\
\dots & \Omega^{(-1|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|s)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|s)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|s)} & \xrightarrow{\mathrm{d}} & \Omega^{(m+1|s)} & \dots \\
 & \vdots & & \vdots & & & \vdots & & & \vdots & & \vdots & \\
 & Z\uparrow & & Z\uparrow\downarrow Y & & & Z\uparrow\downarrow Y & & & Z\uparrow\downarrow Y & & \downarrow Y & \\
\dots & \Omega^{(-1|n)} & \xrightarrow{\mathrm{d}} & \Omega^{(0|n)} & \xrightarrow{\mathrm{d}} & \dots & \Omega^{(r|n)} & \dots & \xrightarrow{\mathrm{d}} & \Omega^{(m|n)} & \xrightarrow{\mathrm{d}} & 0 &
\end{array}$$

The operators **Y** and **Z** are known as Picture Changing Operators and act vertically in the complexes.

These two operators are d-closed and they are not d-exact. The **Y** operators are elements of the cohomology

$$H^{(0|m)}(\mathcal{M})$$

This implies that given a pseudo form (p|q) and multiplying it by a PCO $Y_i = \theta_i \delta(d\theta_i)$ we have

$$Y_i : H^{(p|q)}(\mathcal{M}) \to H^{(p|q+1)}(\mathcal{M})$$

This observation implies that if there were cohomology in a given space, this can be mapped into a space with another picture. Since the two complexes $\Omega^{(p|0)}(\mathcal{M})$ and $\Omega^{(p|m)}(\mathcal{M})$ are either bounded from below or from above, this means that there is no cohomology below and above.

So, the cohomology is entirely contained into the square bounded by the 0-forms with 0 pictures and from the integral forms with n-form degree and m-picture.

Actions on supermanifolds

$$S = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)}(\Phi, d\Phi; V, \psi) \wedge \mathbb{Y}^{(0|m)}(V, \psi)$$

$\mathcal{M}^{(n|m)}$ **Supermanifold**, which locally is described by a superspace with n bosonic coordinates and m fermionic coordinates

(V^a, ψ^α) Supervielbein of the supermanifold a=1,…,n, α=1,…,m

$\mathcal{L}^{(n|0)}(\Phi, d\Phi; V, \psi)$ Geometric Lagrangian. It is a function of fields, their differentials, and of the supervielbein. It is a **n-superform** (differential superform)

$\mathbb{Y}^{(0|m)}(V, \psi)$ Poincaré dual to the immersion of a bosonic submanifold into the supermanifold, are view as **Picture Changing Operator**.

n: form number m: picture number

$$S = \int_{\mathcal{M}^{(n|m)}} \mathcal{L}^{(n|0)} \wedge \mathbb{Y}^{(0|m)}$$

Choosing a suitable PCO, the geometric action reduces to the component action

$$\mathbb{Y}^{(0|m)}_{space-time}(V, \psi)$$

Choosing a manifest supersymmetric PCO, the geometric action reduces to the superspace action

$$\mathbb{Y}^{(0|m)}_{susy}(V, \psi)$$

$$S = \int_{\mathcal{M}^n_{bos}} [d^n x] \mathcal{L}^{(n)}(\phi, \partial\phi)$$

$$\mathcal{L}^{(n)}(\phi, \partial\phi) = \mathcal{L}^{(n)}(\Phi, \partial\Phi; V, \psi)\Big|_{\theta=0, \psi=0}$$

$$S = \int_{\mathbb{R}^{(n|m)}} [d^n x d^m \theta] \widehat{\mathcal{L}}^{(n)}(\phi, \partial\phi, \theta)$$

Equivalence

$$S = \int_{\mathcal{M}^n_{bos}} [d^n x] \mathcal{L}^{(n)}(\phi, \partial\phi) \quad \longleftrightarrow \quad S = \int_{\mathbb{R}^{(n|m)}} [d^n x d^m \theta] \widehat{\mathcal{L}}^{(n)}(\phi, \partial\phi, \theta)$$

The two actions are equivalent iff

$$d\mathcal{L}^{(n|0)}(\Phi, d\Phi; V, \psi) = 0$$

The action is closed under some conditions (superspace constraints). Note that it is **n-superform**, so its differential is not trivial.

and two different PCO's differ by exact terms

$$\mathbb{Y}^{(0|m)}_{susy}(V, \psi) = \mathbb{Y}^{(0|m)}_{spacetime}(V, \psi) + d\Omega^{(-1|m)}$$

Definition of PCO’s

Definition of the PCO's

Suppose to immerge a bosonic surface into a supermanifold

$$\iota : \mathcal{M}^{(n)} \longrightarrow \mathcal{M}^{(n|m)}$$

in the trivial way: by setting the fermionic coordinates to zero.
Then, its Poincaré dual is

$$\mathbb{Y}^{(0|m)}_{spacetime} = \prod_{\alpha=1}^{m} \theta^\alpha \delta(d\theta^\alpha)$$

1. It is closed
2. It is not exact (so it belongs to a cohomology space)
3. Any variation of the immersion is d-exact

$$\delta \mathbb{Y}^{(0|m)}_{spacetime} = d\Omega^{(-1|m)}$$

Cartan calculus on supermanifolds

Differential

$$d = d\theta^\alpha D_\alpha + (dx^m + \theta\gamma^m d\theta)\partial_m$$

Even/Odd Vector fields:

$$v = v^\alpha D_\alpha + v^m \partial_m$$ with v^α odd/even, v^m even/odd

Contraction and Lie derivatives

Even $\iota_v\,, \qquad \iota_v^2 = 0\,, \qquad \mathcal{L}_v = d\iota_v + \iota_v d$

Odd $\iota_{\tilde{v}}\,, \qquad \iota_{\tilde{v}}^2 \neq 0\,, \qquad \mathcal{L}_{\tilde{v}} = d\iota_{\tilde{v}} - \iota_{\tilde{v}} d$

New differential operators (distribution-like operators acting on the space of forms)

$$\delta(\iota_{\tilde{v}}) = \int_{-\infty}^{\infty} dt\, e^{it\iota_{\tilde{v}}} \qquad \Theta(\iota_{\tilde{v}}) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{e^{it\iota_{\tilde{v}}}}{t+i\epsilon} dt$$

Finally, following string theory suggestion, we can define our PCO Z. According to our notations, it decreases the picture by removing delta functions.

$$Z_D = \Big[d, \Theta(\iota_D)\Big]$$

- it is closed
- it is not exact (Heaviside Theta function is not a distribution with compact support)
- it depends upon an odd vector field D. But any variation of D, implies that Z it is exact
- it is not a derivation with respect to the wedge product of forms
- it acts vertically along the complexes of forms, from integral form to diff. forms
- it can be combined with other PCO's Z as follows

$$Z_{Max} = \prod_{p=0}^{m} Z_{D_p}$$

where the odd vector fields D_p are linearly independent.

Super Chern-Simons and non-associative algebras

Let us now consider the supersymmetric version of CS theory. The general expression is of the following form

$$S_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}\Big(A^{(1|0)} \wedge A^{(1|0)} + \frac{2}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} + \frac{1}{2} W^{(0|0)} \cdot W^{(0|0)} V^{(3|0)}\Big) \wedge \mathbb{Y}^{(0|2)}$$

- The gauge connection is replaced by a (1|0) form.
- The gauge group is still a bosonic group, the gauge connection is Lie-algebra valued.
- The integral is over the full supermanifold according to the discussion above.
- $\mathbb{Y}^{(0|2)}$ is a generic PCO, which transform the action into a integral form.

The function $W^{(0|0)}$ is related to the gauge connection using the Bianchi identities

$$F^{(2|0)} = V^a \wedge V^b F_{[ab]} + V^a (\psi \gamma^b W) \qquad dW^\alpha = V^a \nabla_a W^\alpha - \frac{1}{4} (\gamma^{ab} \psi)^\alpha F_{[ab]}$$

The 3d vielbeins satisfy the following MC equations

$$d\psi = 0 \qquad dV^a = \frac{1}{2} \psi \gamma^a \psi \qquad V^{(3|0)} = \epsilon_{abc} V^a \wedge V^b \wedge V^c$$

• If we simplified to the an abelian gauge group, we drop the interaction term (and the covariant derivative from the Bianchi ids).

$$S^{abelian}_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \left(A^{(1|0)} \wedge dA^{(1|0)} + \frac{1}{2} W^{(0|0)} \cdot W^{(0|0)} V^{(3|0)}\right) \wedge \mathbb{Y}^{(0|2)}$$

• Now, we observe that any (0|2) PCO can be decomposed into a product to two (0|1) PCO (with the correct properties) - up to total derivatives

$$\mathbb{Y}^{(0|2)} = \mathbb{Y}^{(0|1)} \wedge \mathbb{Y}^{(0|1)} + d\Omega^{(-1|2)}$$

• Thus, we can rewrite the action by distributing the PCO's over the fields as follows

$$S^{abelian}_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \left((A^{(1|0)} \wedge \mathbb{Y}^{(0|1)}) \wedge d(A^{(1|0)} \wedge \mathbb{Y}^{(0|1)}) + \frac{1}{2}(W^{(0|0)}\mathbb{Y}^{(0|1)}) \cdot (W^{(0|0)}\mathbb{Y}^{(0|1)}) V^{(3|0)}\right)$$

• Finally, we can rewrite the action as follows (in terms of pseudoforms)

$$S^{abelian}_{SCS} = \int_{\mathcal{SM}^{(3|2)}} \left(A^{(1|1)} \wedge dA^{(1|1)} + \frac{1}{2} W^{(0|1)} \cdot W^{(0|1)} V^{(3|0)}\right)$$

To add the interactions, we need a 2-product with the following property:

$$M_2 : \Omega^{(1|1)} \otimes \Omega^{(1|1)} \longrightarrow \Omega^{(2|1)}$$

This situation has strong analogies with string theory and superstring theory (the ghost number has to be correctly compensated for meaningful actions. In the case of superstrings the picture number has the same role as here. It must be saturated for non-trivial contributions. In the case of g super Riemann surfaces q = 2-2g

Using the PCO Z, $Z_D : \omega^{(p|q)} \longrightarrow \omega^{(p|q-1)}$

Erler, Konopka and Sachs (arXiv:1312.2948) proposed the expression

$$M_2\Big(\omega_1^{(1|1)}, \omega_2^{(1|1)}\Big) = \frac{1}{3}\Big(Z_D(\omega_1^{(1|1)} \wedge \omega_2^{(1|1)}) + Z_D(\omega_1^{(1|1)}) \wedge \omega_2^{(1|1)} + (\omega_1^{(1|1)}) \wedge Z_D(\omega_2^{(1|1)})\Big)$$

in terms of which we have

$$\mathcal{L}^{Int} = \mathrm{Tr}\Big(A^{(1|1)} \wedge M_2(A^{(1|1)}, A^{(1|1)})\Big)$$

The 2-product of EKS is not associative

$$M_2(M_2(A,B),C) + M_2(A, M_2(B,C)) \neq 0$$

If we identify the differential d with the 1-product, with the property

$$dM_2(A,B) = M_2(dA,B) + (-1)^{|A|} M_2(A,dB))$$

it turns out that the 2-product satisfies

$$M_2(M_2(A,B),C) + M_2(A,M_2(B,C)) =$$
$$dM_3(A,B,C) + M_3(dA,B,C) + (-1)^{|A|} M_3(A,dB,C) + (-1)^{|A|+|B|} M_3(A,B,dC)$$

which is the starting relation for an A_∞ algebra.

Every algebraic structure is purely based on differential forms and on the supergeometry I discussed. The A_∞ is extended to the whole complex of forms.

$$S_{SCS} = \int_{\mathcal{SM}^{(3|2)}} Tr\Big(A^{(1|1)} \wedge \sum_{n=1}^{\infty} M_n(A^{(1|1)}, A^{(1|1)})\Big)$$

- We have effectively built a super CS theory with the interactions terms which is gauge invariant and it is written as an integral on a supermanifold.

- The algebraic structures emerging from supergeometry are parallel to those emerging in super string field theory (see Erler, Konopka, Sachs) and we are able to reproduce their results using only the geometrical properties.

- We extended their construction to the complete complex of forms with any number of fermions.

- The algebraic properties of these new sets of forms have been translated into a sheaf theoretical language posing the construction on a very solid ground.