Quantum Fields over Noncommutative Space-Time

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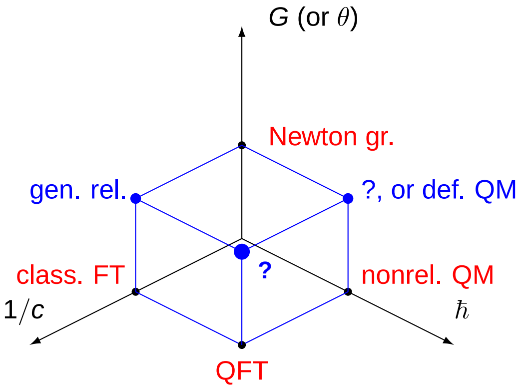
with Raimar Wulkenhaar (and A Sako + A Hock)

COST ACTION MP 1405

Thanks! Richard, working groups,...Peter,...

Limited localisation in space-time: Strings,.........NCG............LQG



History

- 80´ Alain Connes, index th; NCG;
  Fredholm modules; spectral triples; spectral action;...
- 92 HG + J Madore, Fuzzy Sphere regularize 2d QFT
- 93...P Presnajder, C Klimcik modules, supersphere, CP^2...
- 94 DFR canonical deform., Minkowski,

- 1854 Bernhard Riemann
- 1930-38 E Schrödinger, W Heisenberg lattice, divergences of QFT
- 33 R Peierls, 47 H Snyder, C N Yang, 60´ Kadyshevsky Mir-Kasimov,...
- 88 quantum group V Drinfeld, M Jimbo,.... S Majid...
- 90... formulation of quantum spaces, quantum fields,
  J Wess B Zumino,...

- 99 T Krajewski, R Wulkenhaar; V Schomerus; S Minwalla M van Ramsdonk N Seiberg; N Seiberg E Witten

"Summary"

- Matrix Models: P Presnajder, S Kovacik, V Galikova, J Tekel; F Gaetano; H Steinacker; D O´Connor, B Dolan; R Gurau, V Rivasseau;...
- Quantum Spaces: A Sitarz; M Arzano; S Majid; B Jurco; V Dobrev; G Landi, L Dabrowski; P Aschieri; J Pullman; S Paycha;
- Spectral Action: F Lizzi, F D´Andrea, C Stefan, P Martinetti, W van Suijlekom; J Barrett, L Glaser; A Chamseddine,...
- Fields on deformed ST: F Lizzi, P Vitale; R Szabo; D Bahns; J C Wallet; A Cattaneo; J Trampetic, C Martin, J You; M Buric, M Dimitrijevic, M Vojinovic; R Wulkenhaar;...extra dim: G Zoupanos, H Steinacker;
- LQG: C Rovelli; J Lewandowski;... GFT E Wilson-Ewing;
- tensor models: V Rivasseau, R Gurau; N Sasakura; S Y K model
- STRINGS: O Lechtenfeld; R Dawid; S Ferarra; C Gomez; L Jonke; K Wendland; P Grassi; J Lukierski; A Chatzistavrakidis; T Erler; L Thorlacius;... higher structures: C Meusburger; R Szabo, P Schupp; P Severa; ...
- M: E Castellani; C Brukner; P Horava; S B Giddings; R Penrose; M Asorey; CMB S Sarkar; P Traina;

History QFT

- 40´ Viktor Weisskopf, Hans Bethe, E LAMB, E Stückelberg, S Tomonaga, J Schwinger, R Feynman, F Dyson
- 54 L Landau !!! C N Yang-R Mills, W Pauli,...
- 55...A Wightman L Gårding,... regularity, covariance, stability, locality, unitarity, clustering...CPT; Spin-Statistics
- 50´ Wolfgang Pauli: gravity might regularize, R Feynman
- 67 Sakharov induced gravity, Wheeler DeWitt
- 70´ K Symanzik, t´Hooft, Glimm-Jaffe, OS; EW; SUSY SUGRA, Strings, RG, asym.freedom, Glimm-Jaffe
- 78 S Weinberg asymptotic safety, M Reuter,...SM
- 85 M H Goroff A Sagnotti: two loop counterterm
- CDT: R Loll J Ambjørn J Jurkiewicz; Regge calculus, Barrett Crane,...

Gravity: Ignore; External; Semiclassical; emergent; ”NC-ST”; ”true QG”.....

Matricial quantum field theory: HG + R Wulkenhaar,...

. . . is the marriage of

1. matrix models for 2D quantum gravity
2. QFT on noncommutative spaces

1. Kontsevich model
   proves Witten's conjecture that hermitean one-matrix model computes intersection numbers of stable cohomology classes on the moduli space of complex curves
2. Space-time should become a noncommutative manifold at short distances.
   - Euclidean scalar field $\phi \in \mathcal{A}$ (noncommutative algebra)

1. Topological dimension 2 from expansion of matrix models into ribbon graphs, i.e. simplicial 2-complexes.
   - dual to triangulations (Φ^3) or quadrangulations (Φ^4) of 2D-surfaces
   - partition function counts them = 2D quantum gravity

2. Dynamical dimension D encoded in spectrum of operator E,

$$D = \inf\{p \in \mathbb{R}_+ \ : \ \mathrm{tr}((1+E)^{-\frac{p}{2}}) < \infty\}$$

   - ignored in 2D quantum gravity
   - highly relevant for renormalisation of matricial QFT

polynomial	finite	super-ren	just ren.	not ren.
Φ^3	$D < 2$	$2[\frac{D}{2}] \in \{2,4\}$	$2[\frac{D}{2}] = 6$	$2[\frac{D}{2}] > 6$
Φ^4	$D < 2$	$2[\frac{D}{2}] = 2$	$2[\frac{D}{2}] = 4$	$2[\frac{D}{2}] > 4$

A toy model for 4D Φ^4 QFT

- Regularise $\phi^4_{2,4}$ on nc Moyal space $\mathcal{M}_\theta$ with critical oscillator potential.
- Is a matrix model, with infinite number of WI´s from action of $U(\infty)$.
- These Ward id's, and the theory of singular integral equations, turn the Schwinger-Dyson eq's into a fixed point problem.

- For $\theta \to \infty$ we prove existence of a solution.
- Surprisingly, $\theta \to \infty$ describes Schwinger functions on $\mathbb{R}^4$!
- Satisfy (OS1) boundedness, (OS2) invariance (OS4) symmetry.
- Numerics shows evidence for phase transitions.
- ϕ^4_2 "solved" Eric Panzer-Raimar Wulkenhaar, (Lambert)...
- $\phi^3_{2,4,6}$ "solved" top. recursions, g, B, N

Moyal space

algebra of rapidly decaying functions over D-dimensional Euclidean space with $\star$-product

$$(a\star b)(x)=\int d^Dy d^Dk a(x+\tfrac{1}{2}\Theta\cdot k)b(x+y)\,e^{iky}$$

where $\Theta=-\Theta^T\in M_D(\mathbb{R})$

We [H. G, R Wulkenhaar] developed the techniques for:

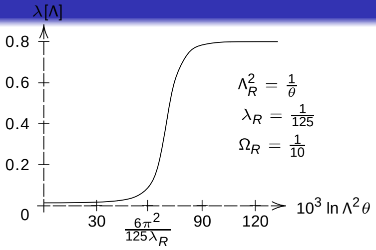
$$S(\phi)=\int_{\mathbb{R}^4} d\xi\Big(\frac{Z}{2}\phi\star(-\Delta+\|\Theta^{-1}\xi\|^2+\mu_{bare}^2)\star\phi+\frac{Z^2\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(\xi)$$

$$=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2} ZE_{\underline{m}}\,\Phi_{\underline{mn}}\Phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)$$

where $E_{\underline{m}}=\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu_{bare}^2}{2}$, $|\underline{m}|:=m_1+m_2\leq\mathcal{N}$, $\quad V=(\frac{\theta}{4})^2$

is renormalizable (Z and μ_{bare} are divergent) and.....

The β-function



one loop

$$\Lambda\frac{d\lambda}{d\Lambda} = \beta_\lambda = c_1\lambda^2(1-\Omega^2) + \mathcal{O}(\lambda^3)$$

$$\Lambda\frac{d\Omega}{d\Lambda} = \beta_\Omega = c_2\lambda(1-\Omega^2) + \mathcal{O}(\lambda^2)$$

$\lambda[\Lambda]$ diverges in commutative case

- perturbation theory remains valid at all scales (duality)! Paris group
- non-perturbative construction done, $\beta = 0$

How does this work?

- four-point function renormalisation with usual sign
- $\exists$ wavefunction renormalisation compensates four-point function renormalisation for $\Omega \to 1$

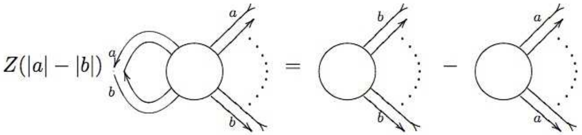
Landau ghost tamed

Ward identity

inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, are not a symmetry of the action, but invariance of measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives:

The insertion of a special vertex

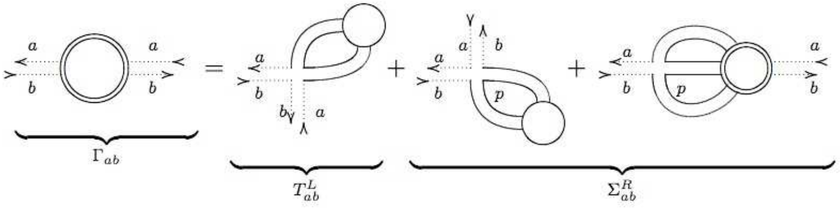
$V^{ins}_{ab} := \sum_n (E_{an} - E_{nb})\phi_{bn}\phi_{na}$ into an external face of a ribbon graph is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$



The dots stand for the remaining face indices.

$$Z(|a| - |b|)G^{ins}_{[ab]\dots} = \quad G_{b\dots} \quad - \quad G_{a\dots}$$

SD equation 2



- vertex is $Z^2\lambda$, connected two-point function is G_{ab}:

$$G_{ab} = H_{ab}^{-1}\Big(1 - Z^2\lambda G_{ab}\sum_q G_{aq} - Z\lambda\sum_p \frac{G_{bp}-G_{ba}}{|p|-|a|}\Big)\,.$$

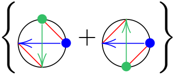
express SD equation in terms of the 1PI function Γ_{ab},
perform nonperturbative renormalisation for the 1PI part,

1PI four-point function

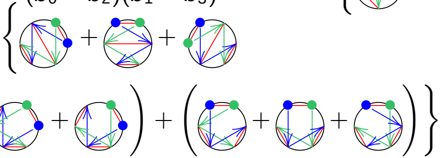
$$\Gamma^{ren}_{abcd} = Z\lambda\Big\{\frac{G^{-1}_{ad}-G^{-1}_{cd}}{|a|-|c|} + \sum_p \frac{G_{pb}}{|a|-|p|}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd}-\Gamma^{ren}_{abcd}\Big)\Big\}$$

Graphical realisation

$$G^{(0)}_{|b_0b_1b_2b_3|} = (-\lambda)\frac{G^{(0)}_{|b_0b_1|}G^{(0)}_{|b_2b_3|} - G^{(0)}_{|b_0b_3|}G^{(0)}_{|b_2b_1|}}{(b_0-b_2)(b_1-b_3)} = -\lambda\Big\{ \cdots + \cdots \Big\}$$



$$G^{(0)}_{|b_0\dots b_5|} = \lambda^2\Big\{ \cdots \Big\}$$



b_i —— b_j $= G^{(0)}_{|b_ib_j|}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

b_i ⟶ b_j $= \frac{1}{b_i - b_j}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Do limit, N and θ go to ∞, ratio fixed:

Solution-Consistency equation

Theorem

$$G_{ab} = \frac{\sin(\vartheta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\vartheta_b] - \mathcal{H}_0[\vartheta_0]} = \frac{e^{\mathcal{H}_a[\vartheta_b(\bullet)] - \mathcal{H}_0[\vartheta_0(\bullet)]}}{\sqrt{(\lambda\pi a)^2 + \left(b + \frac{1+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}\right)^2}}$$

Carleman computes G_{ab}, in particular G_{0b} symmetry forces $G_{b0} = G_{0b}$

Integral equation

Theory determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Assume: Selfconsistency equation for $G^{(0)}_{|ab|}$ has a finite solution after affine renormalisation $E \mapsto Z(E + C\mathbb{1})$ and $\lambda \mapsto Z^2\lambda$. Then: All higher functions $G^{(0)}_{|b_0 \ldots b_{N-1}|}$ with $N \geq 4$ are automatically finite without further need of a renormalisation of λ.

For $g = 0, B = 1$, all N only

S-D equs $\Phi^3_{2,4,6}$ H G + (A Hock) + A Sako + R W

action $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + \alpha\Phi + \frac{\lambda}{3}\Phi^3)$

Strategy

1. use $\mathcal{Z}(J) = K \exp\big(-\frac{\lambda}{3V^2}\sum_{k,l,m=0}^{\mathcal{N}-1}\frac{\partial^3}{\partial J_{kl}\partial J_{lm}\partial J_{mk}}\big)\mathcal{Z}_{free}(J)$ (*)
   $\mathcal{Z}_{free}(J) = \exp\big(\frac{V}{2}\sum_{m,n=0}^{\mathcal{N}-1}\frac{(\alpha\delta_{mn}+J_{mn})(\alpha\delta_{mn}+J_{nm})}{E_m+E_n}\big)$
   to derive equations between the formal power series $G_{\dots}$
2. forget $\mathcal{Z}$, declare equations as exact
3. find non-perturbative solution $G_{\dots}$ of equations

Inserting (*) into $G_{|a|} \equiv \frac{1}{V}\frac{\partial \log Z(J)}{\partial J_{aa}}\Big|_{J=0}$ gives

$$G_{|a|} = \frac{1}{2E_a}\Big(\alpha - \lambda G^2_{|a|} - \frac{\lambda}{V}\sum_{m=0}^{\mathcal{N}-1} G_{|am|} - \frac{\lambda}{V^2}G_{|a|a|}\Big)$$

- the equation is non-linear, making its solution difficult
- $\sum_{m=0}^{\infty} G_{|am|}$ diverges. Choose α such that $G_{|0|} = 0$:

Ward-Takahashi identity

Same computation gives for $N = 2$ and $a \neq b$:

$$G_{|ab|} = \frac{1}{E_a+E_b}\Big(1-\frac{\lambda}{V^2\mathcal{Z}(0)}\sum_{m=0}^{\mathcal{N}-1}\frac{\partial}{\partial J_{ab}}\frac{\partial}{\partial J_{bm}}\frac{\partial}{\partial J_{ma}}\mathcal{Z}(J)\Big|_{J=0}\Big) \qquad (*)$$

WT-Identity

$$\sum_m \frac{\partial^2\mathcal{Z}(J)}{\partial J_{bm}\partial J_{ma}} = \delta_{ab}\mathrm{W}_a(J) + \frac{V}{E_a-E_b}\sum_m\Big(J_{am}\frac{\partial\mathcal{Z}[J]}{\partial J_{bm}} - J_{mb}\frac{\partial\mathcal{Z}[J]}{\partial J_{ma}}\Big)$$

$$\int d\Phi\, e^{-S(\Phi)+V\,\mathrm{tr}(\Phi J)} = \int d\Psi\, e^{-S(\Psi)+V\,\mathrm{tr}(\Psi J)}, \quad \Psi = U^*\Phi U$$

□

inserted into (*): $$G_{|ab|} = \frac{1}{E_a+E_b}\Big(1+\lambda\frac{G_{|a|}-G_{|b|}}{E_a-E_b}\Big)$$

Inserting $G_{|ab|}$ into formula on previous slide gives non-linear equation for $G_{|a|}$ alone, up to $\frac{1}{V^2}(G_{|a|a|} - G_{|0|0|})$ corrections which vanish for $V \to \infty$

Summary: All N-point functions with $B = 1$

Introduce $W_{|a|} := 2(\lambda G_{|a|} + E_a)$. Then:

1. $W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{V^2}\left(G_{|a|a|} - G_{|0|0|}\right) - \frac{2\lambda^2}{V}\sum_{m=0}^{\mathcal{N}-1}\left(\frac{W_{|a|} - W_{|m|}}{E_a^2 - E_m^2} - \frac{W_{|0|} - W_{|m|}}{E_0^2 - E_m^2}\right)$
2. $G_{|ab|} = \frac{1}{2}\frac{W_{|a|} - W_{|b|}}{E_a^2 - E_b^2}$
3. $G_{|a_1 \dots a_N|} = \lambda \frac{G_{|a_1 a_3 \dots a_N|} - G_{|a_2 a_3 \dots a_N|}}{(E_{a_1}^2 - E_{a_2}^2)}$

Proposition: Solution of (2)+(3)

$$G_{|a_1 a_2, \dots a_N|} = \frac{\lambda^{N-2}}{2}\sum_{k=1}^{N} W_{|a_k|}\prod_{l=1, l\neq k}^{N}\frac{1}{E_{a_k}^2 - E_{a_l}^2}$$

Remains to solve (1) for $W_{|a|}$, which is possible in a combined limit $(V, \mathcal{N}) \to \infty$ with $\frac{\mathcal{N}}{V}$ fixed. This eliminates $(G_{|a|a|} - G_{|0|0|})$ and produces a non-linear integral equation for $W_{|a|}$ alone.

The non-linear integral equation and its solution

Assume $E_n = \mu^2(\frac{1}{2} + e(\frac{n}{\mu^2 V}))$ for increasing C^1-function with $e(0) = 0$. In limit $(V, \mathcal{N}) \to \infty$ with $\frac{\mathcal{N}}{V} = \mu^2 \Lambda^2$, after a change of variables $X := (2e(x) + 1)^2$, we have

$$W^2(X) + \int_1^{\Xi} dY \rho(Y) \frac{W(X) - W(Y)}{X - Y} = X + \int_1^{\Xi} dY \rho(Y) \frac{W(1) - W(Y)}{1 - Y} \qquad (*)$$

where $\rho(Y) := \frac{2\lambda^2}{\sqrt{Y} \cdot e'(e^{-1}(\frac{\sqrt{Y}-1}{2}))}$ and $\Xi := (1 + 2e(\Lambda^2))^2$

Theorem (inspired by Makeenko-Semenoff)

The non-linear integral equation (*) is solved by

$$W(X) := \sqrt{X + c} + \frac{1}{2} \int_1^{\Xi} dY \frac{\rho(Y)}{(\sqrt{X + c} + \sqrt{Y + c})\sqrt{Y + c}}$$

where $c(\lambda, e)$ is the inverse solution of

$$W(1) = 1 = \sqrt{1 + c} + \frac{1}{2} \int_1^{\Xi} dY \frac{\rho(Y)}{(\sqrt{1 + c} + \sqrt{Y + c})\sqrt{Y + c}}$$

Solution...

The (1+1)-point function is given by

$$G(X|Y) = \frac{4\tilde{\lambda}^2}{\sqrt{X+c}\cdot\sqrt{Y+c}\cdot(\sqrt{X+c}+\sqrt{Y+c})^2}, \qquad (1)$$

Solution for the (1+1+1)-point function (factorises):

$$G(X|Y|Z) = \frac{(-32)\gamma\tilde{\lambda}^5}{\sqrt{X+c}^3\sqrt{Y+c}^3\sqrt{Z+c}^3}. \qquad (2)$$

Solution for the $(1+\ldots+1)$-point function for $B \geq 4$

$(1+\cdots+1)$-point function with $B \geq 3$

$$G(X^1|\ldots|X^B) = (-2\tilde{\lambda})^{3B-4}\frac{d^{B-3}}{dt^{B-3}}\left(\frac{\frac{1}{\sqrt{X^1+c-2t}^3}\cdots\frac{1}{\sqrt{X^B+c-2t}^3}}{\left(1-\int_1^\infty\frac{dT\,\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\right)^{B-2}}\right)\Bigg|_{t=0} \qquad (3)$$

$$G(X^1|\ldots|X^B) = \frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0}\sum_{M=0}^{B-3}\gamma_B^M\frac{d^M}{dt^M}\sqrt{X+c-2t}^{-3}_{\{1,\ldots,B\}}\Big|_{t=0},$$

Coefficients γ_B^M are Bell polynomials...g=0; higher g: top. recursion

2 pt fct: OS

- Reflection positivity gives spectrum condition which guarantees representation as Laplace transform in ξ^0, hence analyticity in $\mathrm{Re}(\xi^0) > 0$.

Proposition

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}$$

with ρ positive and non-decreasing. Proof: Källén-Lehmann

- Stieltjes property: numerical checks show no violation...

Is positivity in quantum field theory
(Hilbert space scalar product and spectrum condition)

An analogy

2D Ising model	4D nc ϕ^4-theory
temperature T, $K = \frac{J}{k_B T}$	frequency Ω
Kramers-Wannier duality $\sinh(2K)\sinh(2K^*) = 1$	Langmann-Szabo duality $\Omega\Omega^* = 1$
solvable at $K = K^*$ scale-invariant	solvable at $\Omega = \Omega^*$ almost scale-invariant
CFT minimal model	matrix model
operator product expansion Virasoro constraints	Schwinger-Dyson equation Ward identities
critical exponents $G^{\sigma\sigma}_{n0} \propto \frac{1}{n^{d-2+\eta}}$, $\eta = \frac{1}{4}$	critical exponents $G^{\phi\phi}_{n0} \propto \frac{1}{n^{1+\eta(\lambda)}}$, $\lambda \in]\lambda_c, 0]$
Virasoro algebra, CFT, subfactors, ...	Vir alg; top recursion; ?

Euclidean ϕ^3 model (all g) and $\phi^4, g = 0$ model well understood.....

Thank You Richard

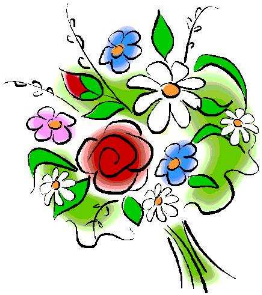
G Zoupanos, F Lizzi

B Jurco, M Schlichenmaier,
J Pulmann

H Steinacker, J Barrett,
C Meusburger

O Lechtenfeld, L Jonke, P Vitale

M Sakellariadou, P Schupp,
W v Suijlekom



To clarify OS another COST program is needed