Fluxes, Dualities and para-Hermitian Geometry

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February 2019

Bratislava '19

based on joint work with R. Szabo arXiv:1810.03953

Motivation

- Global formulation of Double Field Theory [Vaisman, Freidel, Rudolph, Svoboda].
- Clear relationship with Generalized Geometry [Freidel, Rudolph, Svoboda].
- Description of geometric and non-geometric backgrounds in string theory [Hull, Reid-Edwards].

Outline

Almost Para-Hermitian Structures

Definition

An almost para-Hermitian structure (M, K, η) is given by an $2n$-dimensional manifold M, an almost para-complex structure $K \in \mathrm{End}(TM)$ (s.t. $K^2 = \mathbb{1}$) and a metric η with (n, n) signature satisfying the compatibility condition

$$\eta(K(X), Y) + \eta(X, K(Y)) = 0$$

Properties:

- The eigenbundles L_+, L_- are maximally isotropic w.r.t. η, i.e. $\eta(X, Y) = 0, \ \forall X, Y \in \Gamma(L_+) \text{ and } \forall X, Y \in \Gamma(L_-)$.
- The compatibility condition induces the definition of the *fundamental 2-form* $\omega(X, Y) = \eta(K(X), Y)$.

Example: Cotangent Bundles

Consider $\pi : T^*Q \to Q$, then we have

$$0 \to V \to T(T^*Q) \to \pi^*(TQ) \to 0$$

where $V = \mathrm{Ker}(\mathrm{d}\pi)$.

- A splitting C of the sequence defines $T(T^*Q) = V \oplus H_C$, with $\Gamma(H_C) = \mathrm{Span}_{C^\infty(M)}\{h_i = \partial_i + C_{ij}\tilde{\partial}^j \in \Gamma(T(T^*Q))\}$.
- K_C defined by $K_C(\Gamma(H_C)) = \Gamma(H_C)$ and $K_C(\Gamma(V)) = -\Gamma(V)$.
- K_C is compatible with the canonical symplectic 2-form ω_0 iff C is symmetric, i.e. there exists a metric η_C s.t. the compatibility condition is satisfied.

Example: Phase Space Dynamics

$Q = \mathbb{R}^3$: configuration space of a charged particle moving in a magnetic field $\mathcal{B}$ generated by a magnetic charge distribution.

- Almost para-Hermitian structure (T^*Q, K_B, η) :
  Splitting $T(T^*Q) = V \oplus H_B$ such that $V = Ker(\mathrm{d}\pi)$ and $\Gamma(H_B) = \mathrm{Span}_{C^\infty(M)}\{h_i = \partial_i - \epsilon_{ijk}\mathcal{B}^j\tilde{\partial}^k \in \Gamma(T(T^*Q))\}$.
- K_B defined by $K_B(\Gamma(H)) = \Gamma(H)$ and $K_B(\Gamma(V)) = -\Gamma(V)$.
- *Define* a flat η such that $\eta(\Gamma(H), \Gamma(H)) = \eta(\Gamma(V), \Gamma(V)) = 0$.
- Almost symplectic 2-form $\omega_B = dq^i \wedge dp_i - 2\epsilon_{ijk}\mathcal{B}^k dq^i \wedge dq^j$, gives the magnetic Poisson brackets.

Para-Hermitian Connections

Definition

A para-Hermitian connection $\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M)$ on an almost para-Hermitian manifold (M, K, η) is a connection which preserves both η and K, i.e. $\nabla K = \nabla \eta = 0$.

The Levi-Civita connection of η is para-Hermitian iff $\omega = \eta K$ is symplectic, i.e. iff (M, K, η) is almost para-Kähler.

An almost para-Hermitian manifold with Levi-Civita connection $\mathring{\nabla}$ can always be endowed with a *canonical* para-Hermitian connection given by

$$\nabla^c = P_+ \mathring{\nabla} P_+ + P_- \mathring{\nabla} P_- ,$$

where $P_\pm = \frac{1}{2}(\mathbb{1} \pm K)$.

D-Bracket and Fluxes

Definition

The Canonical D-Bracket on an almost para-Hermitian manifold (M, K, η) *compatible* with K is:

$$\eta([X, Y]^D_K, Z) = \eta(\nabla^c_X Y - \nabla^c_Y X, Z) + \eta(\nabla^c_Z X, Y)$$

such that $[\Gamma(L_\pm), \Gamma(L_\pm)]^D_K \subset \Gamma(L_\pm)$.

Suppose (K, η) and (K', η') are almost para-Hermitian structures on M, then K' is D-integrable w.r.t. K if

$$[\Gamma(L'_\pm), \Gamma(L'_\pm)]^D_K \subset \Gamma(L'_\pm).$$

Fluxes measure D-integrability (or lack of) of a para-Hermitian structure with respect to another one.

Deformation of Para-Hermitian Structures

Definition

A B_+-transformation of an almost para-Hermitian structure (K, η) is an endomorphism of $TM = L_+ \oplus L_-$ given by

$$e^{B_+} = \begin{pmatrix} \mathbb{1} & 0 \\ B_+ & \mathbb{1} \end{pmatrix} \in \mathrm{O}(n, n), \tag{1}$$

where $B_+ : \Gamma(L_+) \to \Gamma(L_-)$ and is such that

$$\eta(B_+(X), Y) = -\eta(X, B_+(Y)) = b_+(X, Y).$$

This induces a transformation of the almost product structure:

$$K \to K_{B_+} = e^{B_+} K e^{-B_+}$$

s.t. (M, η, K_{B_+}) is another almost para-Hermitian structure with $\omega_{K_{B_+}} = \eta K_{B_+} = \omega + 2b_+$.

Example: Back to Phase Space Dynamics

- Para-Kähler structure on T^*Q : $K_0 = \partial_i \otimes dq^i - \tilde{\partial}^i \otimes dp_i$ and $\eta = \mathrm{d}q^i \otimes \mathrm{d}p_i + \mathrm{d}p_i \otimes \mathrm{d}q_i$ given by the choice of the "zero connection", i.e. $C_{ij} = 0$. ω_0 is the fundamental 2-form.

- K_B is obtained from K_0 via a B-transformation defined by $B_+ = \epsilon_{ijk}\mathcal{B}^i\tilde{\partial}^k \otimes dq^j$.

- Almost symplectic structure: $\omega_B = \omega_0 + 2b_+$ with $b_+ = \eta B_+ = \epsilon_{ijk}\mathcal{B}^j dq^i \wedge dq^k$ and $\mathrm{d}\omega_B = 2\mathrm{d}b_+$.

- D-Bracket and magnetic charge density:

$$[h_i, h_j]^D_{K_0} = \partial_i(\epsilon_{jkl}\mathcal{B}^l)\tilde{\partial}^k = \eta^{-1}(\mathrm{d}b_+(h_i, h_j))$$

Doubled Twisted Torus

$M = T^*\mathrm{H}/\Lambda$, where H is the (3-dim.) Heisenberg group, $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$ and Λ is a discrete cocompact subgroup of $T^*\mathrm{H}$.

The para-Hermitian structure is derived from the left-invariant one on the Drinfel'd Double $T^*\mathrm{H}$, i.e. $T(T^*\mathrm{H}) \cong T^*\mathrm{H} \times (\mathfrak{h} \ltimes \mathbb{R}^3)$ s.t. $K(Z_i) = Z_i$ and $K(\tilde{Z}^i) = -\tilde{Z}^i$, where $\{Z^i, \tilde{Z}^i\}$ is a basis of left-invariant vector fields.

The splitting of TM is induced by the one of left-invariant vector fields on $T^*\mathrm{H}$.

The metric η is obtained from the duality pairing between $\mathfrak{h}$ and $\mathbb{R}^3$.

Born geometry: Riemannian metric $\mathcal{H}$ inherited from the left-invariant Riemannian metric on $T^*\mathrm{H}$ such that the left-invariant basis spanning $T(T^*\mathrm{H})$ is orthonormal.

Nilmanifold Polarization

The splitting discussed before carries the Lie algebra structure:

$$[Z_x, Z_y] = mZ_z, \quad [Z_x, \tilde{Z}^y] = m\tilde{Z}^z, \quad [Z_z, \tilde{Z}^y] = -m\tilde{Z}^x,$$

with m integer.

$\mathcal{H}$ can be written in terms of the metric

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -mx \\ 0 & -mx & 1+(mx)^2 \end{pmatrix}$$

and the 2-form $b = 0$.

Q-flux: $[Z_x, Z_y]^D_{K_0} = mZ_z$.

NS-NS background with H-flux

There exists a B-transformation which maps the previous splitting into the following one:

$$[Z'_x, Z'_y] = -m\tilde{Z}'^z, \quad [Z'_x, Z'_z] = m\tilde{Z}'^y, \quad [Z'_z, Z'_y] = m\tilde{Z}'^x$$

$\mathcal{H}$ transforms into the Riemannian metric compatible with the new splitting and can be written in terms of $G = diag(1,1,1)$ and $b = mx\mathrm{d}y \wedge \mathrm{d}z$.

This is the T-dual background of the previous one.

H-flux: $[Z'_i, Z'_j]^D_{K_0} = \eta^{-1}(\mathrm{d}b(Z'_i, Z'_j)) = m\ \epsilon_{ijk}\tilde{Z}'^k$.

We can go on and obtain all of the T-dual backgrounds.

Conclusions and Outlook

- Introduction of a new framework for doubled geometry.
- Interpretation of fluxes in terms of D-brackets.
- Natural emergence of fluxes from B-transformations.
- Description of T-duality for the Doubled Twisted Torus.