Coherent quantization of moduli spaces of flat connections

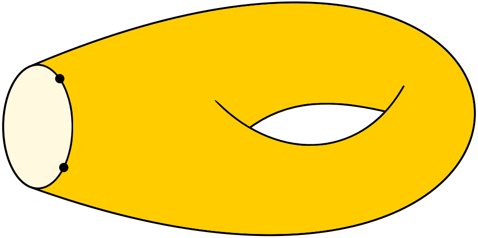
Ján Pulmann

Final QSpace workshop, Bratislava, 2019

Joint work with Pavol Ševera

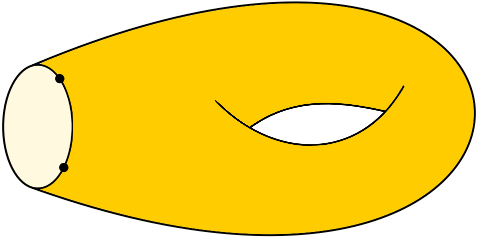
Moduli space of flat connections

- Σ is a surface with boundary, $V \subset \partial\Sigma$ set of points.



Moduli space of flat connections

- Σ is a surface with boundary, $V \subset \partial\Sigma$ set of points.

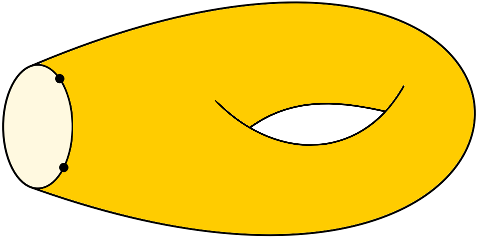


- G is a connected Lie group, $\mathfrak{g}$ the corresponding Lie algebra.
- The moduli space of flat connections

$$\mathcal{M}_{\Sigma,V}(G) = \frac{\{\text{ flat connections on } \Sigma\ \}}{\{g : M \to G \mid g(V) = e\}}$$

Moduli space of flat connections

- Σ is a surface with boundary, $V \subset \partial\Sigma$ set of points.



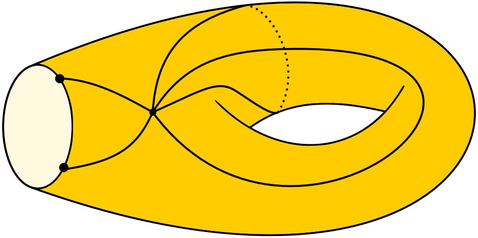
- G is a connected Lie group, $\mathfrak{g}$ the corresponding Lie algebra.
- The moduli space of flat connections

$$\mathcal{M}_{\Sigma,V}(G) = \frac{\{\text{ flat connections on } \Sigma \text{ }\}}{\{g : M \to G \mid g(V) = e\}}$$

- Combinatorial description: enough to specify holonomies along a graph $\Gamma \subset \Sigma$ with $\Gamma \sim \Sigma$

Moduli space of flat connections

- Σ is a surface with boundary, $V \subset \partial\Sigma$ set of points.

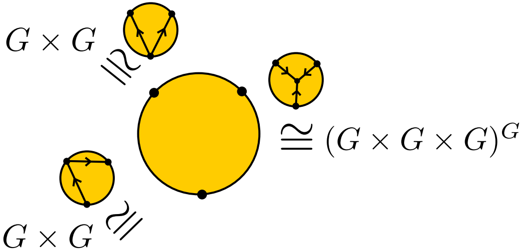


- G is a connected Lie group, $\mathfrak{g}$ the corresponding Lie algebra.
- The moduli space of flat connections

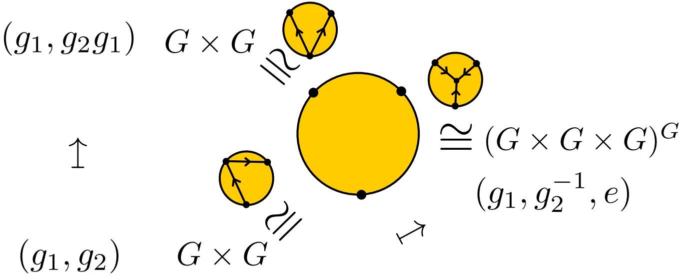
$$\mathcal{M}_{\Sigma,V}(G) = \frac{\{\text{ flat connections on } \Sigma\ \}}{\{g : M \to G \mid g(V) = e\}}$$

- Combinatorial description: enough to specify holonomies along a graph $\Gamma \subset \Sigma$ with $\Gamma \sim \Sigma$

Example

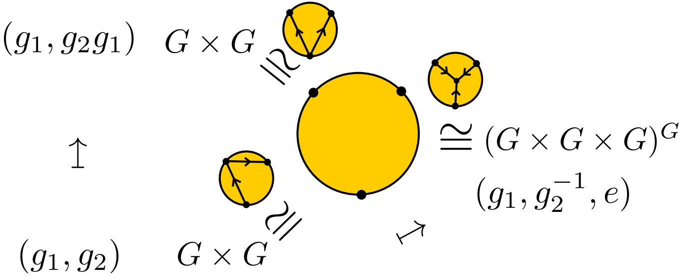


Example

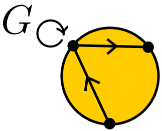


$(g_1, g_2 g_1)$
$G \times G$
$\cong$
$\updownarrow$
$\cong (G \times G \times G)^G$
(g_1, g_2^{-1}, e)
$\mapsto$
(g_1, g_2)
$G \times G$
$\cong$

Example



- There's a residual $G^V = (G \times \cdots \times G)$-action on $M_{\Sigma,V}(G)$, inducing $\mathfrak{g}^V = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ action:



$x \in G : (g_1, g_2) \mapsto (x g_1, g_2 x^{-1})$

Poisson structures on $\mathcal{M}_{\Sigma,V}(G)$

- Let's choose $t \in \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$, e.g. Killing^{-1}.

Theorem (Alekseev, Kosmann-Schwarzbach, Meinrenken)

The moduli space $\mathcal{M}_{\Sigma,V}(G)$ is a $\mathfrak{g}^V$-quasi-Poisson manifold in a canonical way, i.e. it comes with a bivector π such that

$$[\pi, \pi] = (f^{ijk} e_i \wedge e_j \wedge e_k)_{\mathfrak{g}^V}$$

Poisson structures on $\mathcal{M}_{\Sigma,V}(G)$

- Let's choose $t \in \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$, e.g. $\mathrm{Killing}^{-1}$.

Theorem (Alekseev, Kosmann-Schwarzbach, Meinrenken)

The moduli space $\mathcal{M}_{\Sigma,V}(G)$ is a $\mathfrak{g}^V$-quasi-Poisson manifold in a canonical way, i.e. it comes with a bivector π such that

$$[\pi, \pi] = (f^{ijk} e_i \wedge e_j \wedge e_k)_{\mathfrak{g}^V}$$

- Atiyah-Bott: if $\partial\Sigma = \emptyset$, then π comes from a symplectic form.
- if $V = \emptyset$, then RHS is zero and π is Poisson

Deformation quantization

- Find an algebra $\mathcal{A}_\Sigma \underset{\mathrm{Vect}}{\cong} C^\infty(\mathcal{M}_{\Sigma,V}(G))[[\hbar]]$ with a product $\star$ s.t.

$$
\begin{aligned}
a \star b &= a \cdot b + \hbar \dots, \\
(a \star b - b \star a)/\hbar &= \{a, b\} + \hbar \dots
\end{aligned}
$$

Deformation quantization

- Find an algebra $\mathcal{A}_\Sigma \underset{\mathrm{Vect}}{\cong} C^\infty(\mathcal{M}_{\Sigma,V}(G))[[\hbar]]$ with a product $\star$ s.t.

$$a \star b = a \cdot b + \hbar \dots ,$$
$$(a \star b - b \star a)/\hbar = \{a, b\} + \hbar \dots$$

- No Jacobi for $\{,\} \implies$ no associativity for $\star$, but rather:

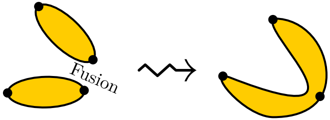
  $(\mathcal{A}_\Sigma, \star)$ is associative in the monoidal category $U(\mathfrak{g}^V)\text{-mod}^\Phi$, where Φ is a Drinfeľd associator, modifying the associativity axiom.

Context

- Previous work: Roche, Szenes; Alekseev, Grosse, Schomerus: quantization via Quantum Groups

Context

- Previous work: Roche, Szenes; Alekseev, Grosse, Schomerus: quantization via Quantum Groups
- Li-Bland, Ševera: given by *fusion* of Σ from discs with 2 marked points $\sim C^\infty(G)$:

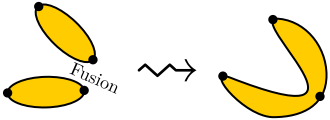


they construct

$$\mathcal{A}_{\Sigma^{\text{unfused}}} \mapsto \mathcal{A}_{\Sigma^{\text{fused}}}$$

Context

- Previous work: Roche, Szenes; Alekseev, Grosse, Schomerus: quantization via Quantum Groups
- Li-Bland, Ševera: given by *fusion* of Σ from discs with 2 marked points $\sim C^\infty(G)$:



they construct

$$\mathcal{A}_{\Sigma^{\text{unfused}}} \mapsto \mathcal{A}_{\Sigma^{\text{fused}}}$$

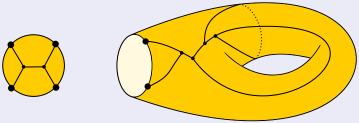
- There are many ways to obtain the same surface by fusion.

How does $\mathcal{A}_\Sigma$ depend on this decomposition?

How does $\mathcal{A}_\Sigma$ depend on this decomposition?

Theorem (P, Ševera)

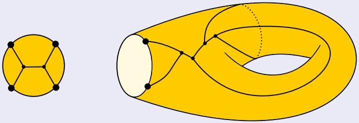
For any uni-trivalent graph $\Gamma \subset \Sigma$ such that univ.vert.$(\Gamma) = V$, there is an algebra $\mathcal{A}_\Gamma$ quantizing $M_{\Sigma,V}(G)$. Examples of Γ:



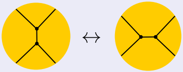
How does $\mathcal{A}_\Sigma$ depend on this decomposition?

Theorem (P, Ševera)

For any uni-trivalent graph $\Gamma \subset \Sigma$ *such that* $\text{univ.vert.}(\Gamma) = V$*, there is an algebra* $\mathcal{A}_\Gamma$ *quantizing* $M_{\Sigma,V}(G)$*. Examples of* Γ*:*



Any two such algebras are canonically isomorphic, the flip move:

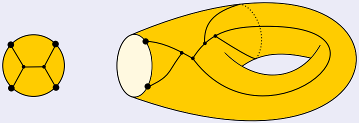


corresponds to the chosen Drinfeld associator.

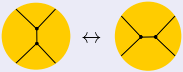
How does $\mathcal{A}_\Sigma$ depend on this decomposition?

Theorem (P, Ševera)

For any uni-trivalent graph $\Gamma \subset \Sigma$ such that univ.vert.$(\Gamma) = V$, there is an algebra $\mathcal{A}_\Gamma$ quantizing $M_{\Sigma,V}(G)$. Examples of Γ:



Any two such algebras are canonically isomorphic, the flip move:

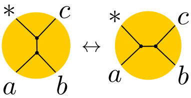


corresponds to the chosen Drinfeľd associator.

Corollary: mapping class group of Σ acts on $\mathcal{A}_\Gamma$

Proof: 2 ideas

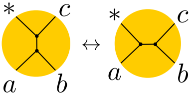
- The trivalent graph Γ encodes “bracketing” of the fusion; the flip move is the rebracketing:



$$(ab)c \leftrightarrow a(bc)$$

Proof: 2 ideas

- The trivalent graph Γ encodes “bracketing” of the fusion; the flip move is the rebracketing:

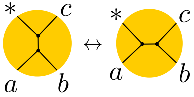


$$(ab)c \leftrightarrow a(bc)$$

- We have to modify fusion of Li-Bland and Ševera using $a \in U(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g}}$: the Kontsevich integral of ⋒

Proof: 2 ideas

- The trivalent graph Γ encodes "bracketing" of the fusion; the flip move is the rebracketing:



$$(ab)c \leftrightarrow a(bc)$$

- We have to modify fusion of Li-Bland and Ševera using $a \in U(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g}}$: the Kontsevich integral of $\Cap$

- WIP: full functoriality under embeddings of surfaces $\rightsquigarrow$ Ševera's quantization of Lie bialgebras.

Thank you!