# Supergravity and Poisson–Lie T-duality

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# Poisson–Lie T-duality

Straightforward robbery from a work of certain Slovak mathematicians.

**Definition**

$(\mathfrak{d}, \mathfrak{g})$ is a **Manin pair** when

- $\mathfrak{d}$ is a quadratic Lie algebra, i.e. equipped with $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ bilinear, non-degenerate, symmetric and invariant;
- $\mathfrak{g} \subseteq \mathfrak{d}$ is a Lagrangian subalgebra, i.e. $\mathfrak{g} = \mathfrak{g}^{\perp}$ w.r.t. $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$;
- $\mathfrak{d} = \mathrm{Lie}(D)$, $\mathfrak{g} = \mathrm{Lie}(G)$ for $G \subseteq D$ closed; both connected.

**Quick recipe for $(g, B, H)$ on $S = D/G$**

1. Pick a maximal positive subspace $\mathcal{E}_{+} \subset \mathfrak{d}$ w.r.t. $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$.
2. Construct a trivial subbundle $\mathcal{E}_{+}^{E} = S \times \mathcal{E}_{+}$ of $E = S \times \mathfrak{d}$.
3. Pick a certain bundle isomorphism $\Psi_{\sigma} : TS \oplus T^{*}S; \rightarrow E$. This gives a unique closed 3-form $H_{\sigma}$.
4. Use $\Psi_{\sigma}$ to define a generalized metric $V_{+}^{\sigma} \subseteq TS \oplus T^{*}S$.
5. $V_{+}^{\sigma}$ uniquely determines metric $g$ and 2-form $B_{\sigma}$ on $S$.

# Supergravity and PLT duality

String low-energy effective action or bosonic part of type II SUGRA with no RR fields is theory for $(g, B, \phi)$ given by:

$$\mathcal{S}_{eff}[g, B, \phi] = \int_S e^{-2\phi}\{\mathcal{R}(g) - \frac{1}{2}\langle H + dB, H + dB\rangle_g + 4\langle d\phi, d\phi\rangle_g\} d\,\mathrm{vol}_g \,. \tag{1}$$

$\phi \in C^\infty(S)$ is called the **dilaton field**.

## Main Question

Is there way to construct $\phi$ in a similar way to $(g, B, H)$ above, such that

- equations of motion (one loop $\beta$-equations) for $(g, B, \phi)$ would depend only on $\mathcal{E}_+$ (and maybe some new data on $\mathfrak{d}$);
- those equations will be independent of the choice of $G \subseteq D$, hence compatible with PLT duality.

Can we found $(g, B, \phi)$ actually solving the equations?

## The answer I (Ševera & Valach, 2018):

Yes, there is.

- more general setting, including RR fields;
- using Courant algebroids, divergencies and GRic;
- examples of solutions only for generalized SUGRA.

Pavol Ševera, Fridrich Valach: **Courant algebroids, Poisson-Lie T-duality, and type II supergravities**, arXiv:1810.07763

## The answer II (Jurčo & Vysoký, 2018):

Yes, there is.

- no RR fields, using Courant algebroids and Levi-Civita connections;
- explicit formulas for $(g, B, \phi)$ and $H$ using the rich geometrical structure of $S = D/G$;
- system of algebraic equations for $\mathcal{E}_+$;
- examples of solutions for ordinary SUGRA.

# Levi-Civita connections

## Definition

Let $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ be a Courant algebroid over $S$. A **Levi-Civita connection** on $E$ with respect to the generalized metric $V_+ \subseteq E$ is a $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, such that

1. $\nabla_{f\psi}(\psi') = f \cdot \nabla_{\psi}(\psi')$
2. $\nabla_{\psi}(f\psi') = f \cdot \nabla_{\psi}(\psi') + \mathcal{L}_{\rho(\psi)}(f) \cdot \psi'$,

where $\nabla_{\psi} \equiv \nabla(\psi, \cdot)$. Moreover, $\nabla_{\psi}(\Gamma(V_+)) \subseteq \Gamma(V_+)$, and $\nabla$ is torsion free, that is

$$0 = \langle \nabla_{\psi}(\psi') - \nabla_{\psi'}(\psi) - [\psi, \psi']_E, \psi'' \rangle_E + \langle \nabla_{\psi''}(\psi), \psi' \rangle_E. \qquad (2)$$

The space of Levi-Civita connections $\mathrm{LC}(E, V_+)$ is very big.

## Definition

Let $\nabla \in \mathrm{LC}(E, V_+)$. Define its **divergence** $\mathrm{div}_{\nabla} : \Gamma(E) \to C^{\infty}(S)$ as $\mathrm{div}_{\nabla}(\psi) = \mathrm{Tr}(\nabla(\cdot, \psi))$. It has the properties of divergence operator from the previous talk.

- There exists a well-defined generalized Riemann tensor $R_\nabla \in \mathcal{T}_4^0(E)$ with satisfactory symmetries for every $\nabla \in \mathrm{LC}(E, V_+)$.
- It allows unambiguous definition of **Ricci tensor** $\mathrm{Ric}_\nabla \in \Gamma(S^2 E^*)$:

$$\mathrm{Ric}_\nabla(\psi, \psi') = \mathrm{Tr}_{g_E}(R_\nabla(\cdot, \psi, \cdot, \psi')), \tag{3}$$

  where $g_E = \langle \cdot, \cdot \rangle_E$. Using generalized metric $V_+ \subseteq E$, we can take its trace to obtain the **scalar curvature** $\mathcal{R}^+_\nabla \in C^\infty(S)$.
- We say that $\nabla$ is **Ricci-compatible** with $V_+$, if $\mathrm{Ric}(V_+, V_-) = 0$.

**Definition**

Let $\phi \in C^\infty(S)$. We consider a subclass $\mathrm{LC}(E, V_+, \phi)$ of Levi-Civita connections satisfying an additional relation for all $\psi \in \Gamma(E)$

$$\mathcal{L}_{\rho(\psi)}(\phi) = \frac{1}{2}(\mathrm{div}_{\nabla^g}(\rho(\psi)) - \mathrm{div}_\nabla(\psi)), \tag{4}$$

where $\nabla^g$ is the L-C connection of $g$ corresponding to $V_+$.

# L-C connections and supergravity

**Theorem (Jurčo & Vysoký)**

*Let $\mathbb{T}S = TS \oplus T^*S$ be equipped with $H$-twisted Dorfman bracket. Let $V_+$ be a generalized metric corresponding to $(g, B)$. Let $\nabla \in \mathrm{LC}(\mathbb{T}S, V_+, \phi)$.*

**Then $(g, B, \phi)$ satisfy the equations of motion given by $\mathcal{S}_{eff}$ if and only if $\mathcal{R}^+_\nabla = 0$ and $\nabla$ is Ricci-compatible with $V_+$.**

*In fact, the action can be written as $\mathcal{S}_{eff}[g, B, \phi] = \int_S e^{-2\phi} \mathcal{R}^+_\nabla \cdot d\,\mathrm{vol}_g$.*

**Theorem**

*Let $\nabla^0 \in \mathrm{LC}(\mathfrak{d}, \mathcal{E}_+)$. Extend it to a connection $\nabla^0_E$ on $E = S \times \mathfrak{d}$. Use the isomorphism $\Psi_\sigma$ to construct $\nabla^\sigma \in \mathrm{LC}(\mathbb{T}S, V^\sigma_+)$. Then*

1. *One has $\mathcal{R}^+_{\nabla^\sigma} = \mathcal{R}^+_{\nabla^0}$ and $\nabla^\sigma$ is Ricci-compatible with $V^\sigma_+$ iff $\nabla^0$ is Ricci-compatible with $\mathcal{E}_+$.*
2. *Everything will work as wanted, if we find $\phi \in C^\infty(S)$, such that $\nabla^\sigma \in \mathrm{LC}(\mathbb{T}S, V^\sigma_+, \phi)$.*

# Geometry of Manin pairs

For a given Manin pair $(\mathfrak{d}, \mathfrak{g})$, there is a rich geometry behind. One can consider the isotropic splittings $j : \mathfrak{g}^* \to \mathfrak{d}$ of the sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{i} \mathfrak{d} \underset{j}{\overset{q}{\rightleftarrows}} \mathfrak{g}^* \longrightarrow 0, \tag{5}$$

where $q$ is the quotient map $\mathfrak{d} \to \mathfrak{d}/\mathfrak{g}$ combined with the identification $\mathfrak{d}/\mathfrak{g} \cong \mathfrak{g}^*$ induced by $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$.

**Objects induced by a choice of $j$**

- $\mu \in \Lambda^3 \mathfrak{g}$, given by $\mu(\xi, \eta, \zeta) = \langle [j(\xi), j(\eta)]_{\mathfrak{d}}, j(\zeta) \rangle_{\mathfrak{d}}$;
- $\mathbb{R}$-bilinear skew-symmetric bracket $[\xi, \eta]_{\mathfrak{g}^*}(x) = \langle [j(\xi), j(\eta)]_{\mathfrak{d}}, i(x) \rangle_{\mathfrak{d}}$;

Not a Lie bracket. There is a bunch of relations among $[\cdot, \cdot]_{\mathfrak{g}}$, $\mu$ and $[\cdot, \cdot]_{\mathfrak{g}^*}$, making $\mathfrak{g}$ into a **Lie quasi-bialgebra**.

There is a canonical action $\triangleright : D \times S \to S$ called the **dressing action** of $D$ on $S$. Let $\#^{\triangleright} : \mathfrak{d} \to \mathfrak{X}(S)$ be the corresponding Lie algebra action.

There is also a **canonical bivector** $\Pi^j_S \in \mathfrak{X}^2(S)$ such that

$$\frac{1}{2}[\Pi^j_S, \Pi^j_S] = \Lambda^3(\#^\triangleright \circ i)(\mu), \;\; \mathcal{L}_{\#^\triangleright(x)} \Pi^j_S = \text{something interesting.} \qquad (6)$$

One says that $j$ **everywhere admissible** if $\#^\triangleright_s \circ j : \mathfrak{g}^* \to T_s S$ is a linear isomorphism for all $s \in S$.

## Properties of such $j$

- Vector fields $\xi^\triangleright = \#^\triangleright(j(\xi))$ globally generate $\mathfrak{X}(S)$.
- One forms $x^\triangleright$ defined by $x^\triangleright(\xi^\triangleright) := \xi(x)$ globally generate $\Omega^1(S)$.
- Define $\mathbf{\Pi} \in C^\infty(S, \Lambda^2 \mathfrak{g}^*)$ as $\mathbf{\Pi}(x,y) = \Pi^j_S(x^\triangleright, y^\triangleright)$. Then

$$\xi^\triangleright(i(x)) = \mathbf{\Pi}(x^\triangleright)^\triangleright. \qquad (7)$$

- Equivalently, the map $\mathbf{k}(d) = pr_{\mathfrak{g}} \circ \mathrm{Ad}_d \circ i \in \mathrm{End}(\mathfrak{g})$ is invertible.
- For each $s_0 \in S$, there exists $U \ni s_0$ and $j$ admissible on $U$.

- Choosing an isotropic splitting $j$, $\mathcal{E}_+$ is uniquelly encoded as a graph of an invertible constant map $E_0 : \mathfrak{g}^* \to \mathfrak{g}$.
- If $\mathfrak{g}$ is a unimodular Lie algebra, that is $\mathrm{Tr}(ad_x) = 0$, the map $\det(\mathbf{k})$ is $G$-invariant and defines a smooth map $\det(\mathbf{k}) : S \to \mathbb{R}$.

## Theorem

*Suppose the following criteria are met:*

1. $\nabla^0 \in \mathrm{LC}(\mathfrak{d}, \mathcal{E}_+)$ *is divergence-free;*
2. $\mathfrak{g}$ *is unimodular, that is* $\mathrm{Tr}(\mathrm{ad}_x) = 0$*;*
3. *there exists an everywhere admissible isotropic splitting* $j$*.*

*Then there exists (unique up to a constant)* $\phi \in C^\infty(S)$ *and isomorphism* $\Psi_\sigma$*, such that* $\nabla^\sigma \in \mathrm{LC}(\mathbb{T}S, V_+^\sigma, \phi)$*, given by formula*

$$\phi = -\frac{1}{2} \ln(\det(\mathbf{1}_{\mathfrak{g}} - E_0 \mathbf{\Pi})) - \frac{1}{2} \ln(\det(\mathbf{k})) \tag{8}$$

*In fact,* $\phi$ *does not depend on* $j$*. As admissible splittings exist locally, this is a local form of a global function in the general case.*

- The 3-form $H_\sigma$ can be written as $H_\sigma(\xi^\triangleright, \eta^\triangleright, \zeta^\triangleright) = \mu(\xi, \eta, \zeta)$.
- Explicit formulas can be written also for $g$ and $B_\sigma$, namely

$$(g + B_\sigma)(\xi^\triangleright, \eta^\triangleright) = \langle \xi, (E_0^{-1} - \mathbf{\Pi})^{-1}(\eta) \rangle. \tag{9}$$

  This boils down to original expressions in PLT duality papers, where $S \cong G^*$, $\mu = 0$, $\xi^\triangleright$ are right-invariant fields on $G^*$ and $\mathbf{\Pi} = -\mathbf{\Pi}^*$.
- One can write $E_0 = g_0^{-1} + \theta_j$, where $g_0 \in S^2(\mathfrak{g}^*)$ and $\theta_j \in \Lambda^2 \mathfrak{g}$.

**Theorem**

*The equations of motion for $(g, B_\sigma, \phi)$ can be rewritten as a system of algebraic equations for $\mathcal{E}_+$ and thus for $g_0$ and $\theta_j$.*
*For a given Manin pair $(\mathfrak{d}, \mathfrak{g})$, the isotropic splitting $j$ can be chosen conveniently to control $\mu$ and $[\cdot, \cdot]_{\mathfrak{g}^*}$.*

**Important question**

Has the system for $\mathcal{E}_+$ any solutions whatsoever? If not, the whole method is a failure.

# Example

- Choose $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, where $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is Abelian.
- Fix the isotropic splitting $j(\xi) = (0, \xi)$.
- Assume that with respect to this splitting, $\mu = 0$ and $[\cdot,\cdot]_{\mathfrak{g}^*}$ is the 3-dimensional Heisenberg Lie algebra on $\mathfrak{g}^*$. In fact, this choice renders $j$ everywhere admissible.
- The integrating $(D, G)$ is $D = \mathfrak{g} \rtimes H_3(\mathbb{R})$, $S \cong H_3(\mathbb{R})$.

## Solution

There exists a solution for $\mathcal{E}_+$, if we relax the positivity condition (this is in fact relevant for physics). In standard $H^3(\mathbb{R})$ coordinates $(\alpha^1, \alpha^2, \alpha^3)$, we plug the solution into the formulas and obtain a metric

$$g = \pm d\alpha^2 \otimes d\alpha^2 + d\alpha^1 \otimes d\alpha^3 + d\alpha^3 \otimes d\alpha^1 - \alpha_3(d\alpha^2 \otimes d\alpha^3 + d\alpha^3 \otimes d\alpha^2)$$

One has $H_\sigma = 0$ and $B_\sigma$ can be arbitrary right-invariant. Dilaton formula gives $\phi = 0$. Not very surprising as $dB_\sigma = 0$ and $g$ is flat.

Branislav Jurčo, Jan Vysoký: **Effective actions for $\sigma$-models of Poisson–Lie type**, to appear soon,

Branislav Jurčo, Jan Vysoký: **Poisson-Lie T-duality of String Effective Actions: A New Approach to the Dilaton Puzzle**, arXiv:1708.04079,

Branislav Jurčo, Jan Vysoký: **Courant Algebroid Connections and String Effective Actions**, arXiv:1612.0154.

**Thank you for your attention!**